Solution for *Classical Electrodynamics, David Jackson*

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1 Introduction to Electrostatics

1. (a) In electrostatic equilibrium, the electric field inside a conductor vanishes entirely. Applying *Gauss's law* in differential form $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$, the absence of the electric field $\boldsymbol{E} = 0$ directly implies a zero net charge density $\rho = 0$ within the bulk material. Consequently, any excess charge must be distributed on the surface of the conductor.

   (b) The problem can be analyzed via the principle of superposition, which allows decoupling the effects of external and internal charges under electrostatic equilibrium.

   - **Case I: external charges only.**
     For any closed surface S entirely within the conductor, *Gauss's law* gives
     $$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = 0.$$
     Thus the exterior charges induce no electric field within the cavity.
   - **Case II: interior charges only.**
     Let the cavity contain a charge Q_{int}, with no external charges. Charge conservation dictates that the cavity's inner surface acquires $-Q_{\text{int}}$, forcing the outer surface to carry $+Q_{\text{int}}$. Applying Gauss's law to a surface S' enclosing the entire conductor
     $$\oint_{S'} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q_{\text{total}}}{\epsilon_0} = \frac{Q_{\text{int}}}{\epsilon_0}.$$
     Thus the exterior field is influenced by the induced surface charge due to the interior charge distribution.

   (c) The absence of the electric field $\boldsymbol{E}$ inside conductor dictates that its surface is equipotential. Consequently, the electric field at the surface is normal to the surface.

   To quantify this field, construct a cylindrical Gaussian surface S intersecting the conductor's surface, with its infinitesimal top face(area ΔS) positioned just outside the conductor and bottom face embedded within the conductor's bulk. The cylinder's height is vanishingly small, ensuring negligible flux through the curved sidewalls. Applying *Gauss's theorm*,
   $$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = E\Delta S = \frac{\sigma \Delta S}{\epsilon_0}.$$
   Thus
   $$E = \frac{\sigma}{\epsilon_0}.$$

2. Let $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}'$. Then,
$$\begin{aligned}
\delta^3(\mathbf{x} - \mathbf{x}') &= \lim_{\alpha \to 0} (2\pi)^{-3/2} \alpha^{-3/2} \exp\left[-\frac{1}{2\alpha}\left(\Delta x^2 + \Delta y^2 + \Delta z^2\right)\right] \\
&= \lim_{\alpha \to 0} \prod_{i=1}^{3} (2\pi)^{-1/2} \alpha^{-1/2} \exp\left[-\frac{(\Delta q_i)^2}{2\alpha h_i^2}\right] \\
&= \prod_{i=1}^{3} \left[\delta\left(\frac{\Delta q_i}{h_i}\right)\right] \\
&= \delta(u - u')\delta(v - v')\delta(w - w') \cdot UVW,
\end{aligned}$$
   where $q_1 = u, q_2 = v, q_3 = w$, and $h_1 = U, h_2 = V, h_3 = W$.

3. (a) Suppose $\rho(\mathbf{x}) = A\delta(r-R)$, where A is normalizing constant and satisfies

$$Q = \iiint_{\infty} \rho r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = 4\pi A \int_0^{\infty} \delta(r-R) r^2 \mathrm{d}r = 4\pi R^2 A.$$

Thus, $A = \dfrac{Q}{4\pi R^2}$ and $\boxed{\rho = \dfrac{Q}{4\pi R^2}\delta(r-R)}$

(b) By analogy to (a), $\boxed{\rho = \dfrac{\lambda}{2\pi b}\delta(r-b)}$

(c) Charges are confined to $z=0$ and $r<R$, so the charge densities can be written as the product of a delta function and a Heaviside step function

$$\rho(\mathbf{x}) = A\delta(z)\theta(R-r),$$

where $\theta(x) = \begin{cases} 0, & x<0, \\ \frac{1}{2}, & x=0, \\ 1, & x>0 \end{cases} = \int_{-\infty}^{x} \delta(t)\mathrm{d}t$. Then, the total charge is

$$Q = \iiint_{\infty} \rho r \mathrm{d}r\mathrm{d}\theta\mathrm{d}z = A\int_0^{2\pi}\mathrm{d}\theta\int_{-\infty}^{\infty}\delta(z)\mathrm{d}z\int_0^{\infty}\theta(R-r)r\mathrm{d}r = A\cdot 2\pi\cdot 1\cdot\frac{1}{2}R^2 = \pi R^2 A,$$

which yields $A = \dfrac{Q}{\pi R^2}$ and then $\boxed{\rho = \dfrac{Q}{\pi R^2}\delta(z)\theta(R-r) = \dfrac{Q}{\pi R^2}\delta(z)\int_{-\infty}^{R-r}\delta(t)\mathrm{d}t}$

(d) By analogy to (c), $\boxed{\rho = \dfrac{Q}{\pi R^2}\delta(\theta-\frac{\pi}{2})\theta(R-r) = \dfrac{Q}{\pi R^2}\int_{-\infty}^{R-r}\delta(\theta-\frac{\pi}{2})\delta(t)\mathrm{d}t}$

4. • Conductor:

When $r<a$, $\boldsymbol{E}=0$; when $r>a$, $\boldsymbol{E} = \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$. Thus,

$$\boxed{\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0 r^2}\theta(r-a)\hat{\boldsymbol{r}}}$$

• Uniform charge density:

When $r<a$, $\boldsymbol{E} = \dfrac{Qr}{4\pi\epsilon_0 a^3}\hat{\boldsymbol{r}}$; when $r>a$, $\boldsymbol{E} = \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} = \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$. Consequently,

$$\boxed{\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0 r^2}\left(\frac{r}{a^3}\theta(a-r) + \frac{1}{r^2}\theta(r-a)\right)\hat{\boldsymbol{r}}}$$

• $\rho \propto r^n (n>-3)$:

Suppose $\rho = \rho_0 r^n$, where ρ_0 is the normalizing constant and satisfies

$$Q = \int \rho \mathrm{d}\tau = 4\pi\rho_0\int_0^a r^{n+2}\mathrm{d}r = \frac{4\pi\rho_0}{n+3}a^{n+3}.$$

It gives $\rho = \dfrac{(n+3)Q}{4\pi a^{n+3}} r^n (r<a)$.

When $r<a$, *Gauss's law* gives

$$4\pi r^2 R = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_0^r 4\pi\rho_0 r^{n+2}\mathrm{d}r = \frac{Q}{\epsilon_0}\frac{r^{n+3}}{a^{n+3}},$$

which yields

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0}\frac{r^{n+1}}{a^{n+3}}\hat{\boldsymbol{r}}.$$

When $r > a$, by applying *Gauss's law*, $\boldsymbol{E} = \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$.

In conclusion, the general solution is

$$\boxed{\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0 r^2}\left(\frac{r^{n+1}}{a^{n+3}}\theta(a-r) + \frac{1}{r^2}\theta(r-a)\right)\hat{\boldsymbol{r}}}$$

5. When $r \neq 0$, the charge density is derived from the Poisson equation as

$$\rho(r) = -\epsilon_0\nabla^2\Phi = -\frac{q\alpha^3}{8\pi}e^{-\alpha r}.$$

As $r \to 0$, $\Phi \to \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{r}$, indicating the presence of a point charge at $r = 0$, with the charge density given by $\rho(r) = q\delta(r)$. Combining these results, the charge density distribution of the hydrogen atom is expressed as

$$\boxed{\rho(r) = q\delta(r) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}}$$

If we integral $\rho(r)$ over all space, we obtain the total charge

$$Q = \int_0^\infty \rho(r)4\pi r^2\mathrm{d}r = 4\pi q\int_0^\infty \delta(r)r^2\mathrm{d}r - \frac{q\alpha^3}{8\pi}\int_0^\infty e^{-\alpha r}r^2\mathrm{d}r = q - \frac{q}{2}\Gamma(3) = 0,$$

which implies that the hydrogen atom is electrically neutral in a time-averaged sense.

6. Suppose each capacitor is charged with q.

   (a) *Gauss's law* gives $E = \dfrac{q}{\epsilon_0 A}$, and thus the potential difference between the two sheets is $U = \dfrac{qd}{\epsilon_0 A}$. Consequently, the capacitance is

   $$\boxed{C = \frac{q}{U} = \frac{\epsilon_0 A}{d}}$$

   (b) *Gauss's law* gives $E = \dfrac{q}{2\pi\epsilon_0 r^2}(a < r < b)$, and the potential difference between the two shells is $U = \int_a^b E\mathrm{d}r = \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$. Thus, the capacitance is

   $$\boxed{C = 4\pi\epsilon_0\frac{ab}{b-a}}$$

   (c) *Gauss's law* gives $E = \dfrac{q}{2\pi r L\epsilon_0}$, and thus the potential difference between the two cylinders is $U = \int_a^b E\mathrm{d}r = \dfrac{q}{2\pi\epsilon_0 L}\ln\dfrac{b}{a}$. Consequently, the capacitance is

   $$\boxed{C = 2\pi\epsilon_0 L\left(\ln\frac{b}{a}\right)^{-1}}$$

7. When $d \gg a_1, a_2$, the charge is approximately uniformly distributed on each surface. Assuming the line densities of the conductors are $\lambda, -\lambda$, respectively, the potential difference between the conductors is

$$U = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{d}{a_1} + \frac{\lambda}{2\pi\epsilon_0} \ln \frac{d}{a_2} = \frac{\lambda}{\pi\epsilon_0} \ln \frac{d}{a}.$$

Thus, the capacitance is approximately

$$C \simeq \pi\epsilon_0 \left(\ln \frac{d}{a} \right)^{-1}.$$

8. • Two large, flat, conducting sheets:

(a) The total electrostatic energy stored in the capacitor is

$$W = \frac{Q^2}{2C} = \frac{Q^2 d}{2\epsilon_0 A}.$$

The potential difference between the two sheets is

$$U = \frac{Q}{C} = \frac{Qd}{\epsilon_0 A}.$$

(b) Since the conducting sheets is very large and the distance between them is very small, we can neglect the edge effects. Consequently, we can just assume the total electrostatic energy is stored in the volume between the two sheets, and the electric field is uniform. Therefore, the energy density inside the space between the sheets is

$$\omega = \frac{W}{dA} = \frac{Q^2}{2\epsilon_0 A^2}$$

while $\omega = 0$ elsewhere.

• Two concentric conducting spheres:

(a) The total electrostatic energy stored in the capacitor is

$$W = \frac{Q^2}{2C} = \frac{Q^2(b-a)}{8\pi\epsilon_0 ab}.$$

The potential difference between the two spheres is

$$U = \frac{Q}{C} = \frac{Q(b-a)}{4\pi\epsilon_0 ab}.$$

(b) By applying *Gauss's law*, the distribution of the electric field is given as

$$E = \begin{cases} 0, & r < a, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}, & a < r < b, \\ 0, & r > b. \end{cases}$$

Thus, the energy density can be expressed as

$$\omega = \frac{1}{2}\omega_0 E^2 = \begin{cases} 0, & r < a, \\ \dfrac{Q^2}{32\pi^2\epsilon_0 r^4}, & a < r < b, \\ 0, & r > b. \end{cases}$$

- Two concentric conducting cylinders:

  (a) The total electrostatic energy stored in the capacitor is

  $$W = \frac{Q^2}{2C} = \frac{Q^2}{2\pi\epsilon_0 L}\ln\frac{b}{a}.$$

  The potential difference between the two cylinders is

  $$U = \frac{Q}{C} = \frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a}.$$

  (b) By applying *Gauss's law*, the distribution of the electric field is approximately given as

  $$E = \begin{cases} 0, & r < a, \\ \dfrac{Q/L}{2\pi\epsilon_0 r}, & a < r < b, \\ 0, & r > b. \end{cases}$$

  Thus, the energy density can be expressed as

  $$\omega = \frac{1}{2}\epsilon_0 E^2 = \begin{cases} 0, & r < a, \\ \dfrac{Q^2}{8\pi^2\epsilon_0 r^2 L^2}, & a < r < b, \\ 0, & r > b. \end{cases}$$

9. (a)

- Case 1: two parallel plate

  According to the solution of 8.(a), the total electrostatic energy stored in the capacitor is

  $$W = \frac{Q^2 d}{2\epsilon_0 A}.$$

  If a force F quasi-statically pulls the plates apart by an infinitesimal displacement δd, the work done by the force equals the change in the system's energy:

  $$\delta W = F\delta d = \frac{Q^2}{2\epsilon_0 A}\delta d.$$

  Consequently,

  $$F = \frac{Q^2}{2\epsilon_0 A}.$$

  Under the quasi-static condition, this force F corresponds to the attractive force between the two plates.

- Case 2: two cylindrical conductors

  The total electrostatic energy stored in this capacitor is

  $$W = \frac{Q^2}{2\pi\epsilon_0}\ln\frac{d}{a}.$$

  Similarly, we have

  $$\delta W = F\delta d = \frac{Q^2}{2\pi\epsilon_0}\delta\left(\ln\frac{d}{a}\right) = \frac{Q^2}{2\pi\epsilon_0 d}\delta d.$$

  Therefore,

  $$F = \frac{Q^2}{2\pi\epsilon_0 d}.$$

(b) It's impossible for a isolating conductor to keep the potential difference between the conductors. Thus there must be a electromotive force (*emf*) at the exterior of the capacitor, and $\mathcal{E} = U$. If a force F quasi-statically pulls the two conductors apart by an infinitesimal displacement δd again, the change in the system energy satisfies

$$\delta W = F\delta d + U\delta q,$$

where $q = CU$, and $W = \frac{1}{2}CU^2$. Therefore,

$$F = -\frac{\delta W}{\delta d}.$$

- Case 1: two parallel plane The total electrostatic energy is

$$W = \frac{1}{2}CU^2 = \frac{1}{2}\frac{\epsilon_0 A}{d}U^2.$$

Thus, the attractive force between them is

$$F = -\frac{\delta W}{\delta d} = \frac{1}{2}\frac{\epsilon_0 A}{d^2}U^2.$$

- Case 2: two cylindrical conductors The total electrostatic energy is

$$W = \frac{1}{2}CU^2 = \frac{1}{2}\pi\epsilon_0 U^2\left(\ln\frac{d}{a}\right)^{-1}.$$

Therefore, the attractive force between them is

$$F = -\frac{\delta W}{\delta d} = \frac{\pi\epsilon_0 U^2}{2d}\left(\ln\frac{d}{a}\right)^{-2}.$$

10. Suppose the radius of the sphere is R, and the center is at the point P. The average potential over the surface of this sphere is

$$\langle V\rangle = \frac{1}{4\pi R^2}\oint_{S_R} V\mathrm{d}A.$$

Consequently,

$$\frac{\mathrm{d}\langle V\rangle}{\mathrm{d}R} = \frac{1}{4\pi R^2}\oint_{S_R}\frac{\partial V}{\partial R}\mathrm{d}A + \frac{1}{4\pi R^2}\oint_{S_R} V\frac{\partial \mathrm{d}A}{\partial R} - \frac{2}{4\pi R^3}\oint_{S_R}\phi\mathrm{d}A.$$

- The first term:

$$\frac{1}{4\pi R^2}\oint_{S_R}\frac{\partial\phi}{\partial R}\mathrm{d}A = \frac{1}{4\pi R^2}\oint_{S_R}\nabla V\cdot\mathrm{d}\boldsymbol{S} = \frac{1}{4\pi R^2}\iiint\nabla^2\mathrm{d}\tau.$$

Since there's no charge inside the sphere, $\nabla^2 V = 0$. Therefore, the first term vanishes.

- The second term:

The area element can be expressed as $\mathrm{d}A$ can be written as $\mathrm{d}A = R^2\mathrm{d}\Omega$, where Ω is the solid angle and independent of R. Thus $\frac{\partial \mathrm{d}A}{\mathrm{d}R} = 2R\mathrm{d}\Omega = \frac{2}{R}\mathrm{d}A$. Substituting this into the second term gives

$$\frac{1}{4\pi R^2}\oint_{S_R}\frac{2V}{R}\mathrm{d}A,$$

which cancels the third term.

In conclusion, we obtain

$$\frac{\mathrm{d}\langle V\rangle}{\mathrm{d}R} = 0,$$

which demonstrates that the average of the potential over the surface is independent of R. Taking the limit as $R \to 0$, we have

$$\lim_{R\to 0}\langle V\rangle = V(P).$$

11. Consider a deformed cylindrical Gaussian surface S just outside a conductor, with its top and bottom faces parallel to the conductor's curved surface. The height of the cylinder is infinitesimal, and the bottom face has an area

$$\mathrm{d}A = (R_1\mathrm{d}\theta_1)(R_2\mathrm{d}\theta_2) = R_1R_2\mathrm{d}\theta_1\mathrm{d}\theta_2,$$

where θ_1, θ_2 are the angular variables aligned with each principle radii. The area of the top surface is

$$\mathrm{d}A' = (R_1+\mathrm{d}n)\mathrm{d}\theta_1(R_2+\mathrm{d}n)\mathrm{d}\theta_2 = \left(\frac{R_1+\mathrm{d}n}{R_1}\right)\left(\frac{R_2+\mathrm{d}n}{R_2}\right)\mathrm{d}A = \left(1+\frac{\mathrm{d}n}{R_1}+\frac{\mathrm{d}n}{R_2}\right)\mathrm{d}A,$$

where the higher-order terms are neglected. Since no charge is enclosed within the Gaussian surface, *Gauss's law* gives

$$\begin{aligned}
0 &= \oint_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} \\
&= (E+\mathrm{d}E)\mathrm{d}A' - E\mathrm{d}A \\
&= \left(E+\frac{\partial E}{\partial n}\mathrm{d}n\right)\left(1+\frac{\mathrm{d}n}{R_1}+\frac{\mathrm{d}n}{R_2}\right)\mathrm{d}A' - E\mathrm{d}A \\
&= \left[\frac{\partial E}{\partial n}+E\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]\mathrm{d}n\mathrm{d}A.
\end{aligned}$$

Consequently,

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

12. *Green's theorem* is given by formula (1.35) on page 36 of the main text:

$$\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)\mathrm{d}\tau = \oint_S\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}A.$$

By choosing $\phi=\Phi, \psi=\Phi'$, we obtain

$$\int_V(\Phi\nabla^2\Phi'-\Phi'\nabla^2\Phi)\mathrm{d}\tau = \oint_S\left(\Phi\frac{\partial\Phi'}{\partial n}-\Phi'\frac{\partial\Phi}{\partial n}\right)\mathrm{d}A.$$

Since $\nabla^2\Phi = -\frac{\rho}{\epsilon_0}, \nabla^2\Phi' = -\frac{\rho'}{\epsilon_0}$, and $\frac{\partial\Phi}{\partial n} = \frac{\sigma}{\epsilon_0}, \frac{\partial\Phi'}{\partial n} = \frac{\sigma'}{\epsilon_0}$, by substituting these into the equation above, we have

$$\int_V\Phi\rho'\mathrm{d}\tau + \oint_S\Phi\sigma'\mathrm{d}A = \int_V\Phi'\rho\mathrm{d}\tau + \oint_S\Phi'\sigma\mathrm{d}A.$$

13. Assume the potential of the upper plane is Φ_1, the potential of the lower plane is Φ_2, and the potential of the point charge is Φ. Meanwhile, assume the distance between the point charge and the lower plane is h, and the distance between two planes is d. Then consider the following two case:

	Φ_1	Φ_2	Φ	Q_1	Q_2	Q
I	0	0	?	q_1	q_2	q
II	0	$\frac{\sigma}{\epsilon_0}d$	$\frac{\sigma}{\epsilon_0}h$	σ(uniformly charged)	σ(uniformly charged)	0

Green's reciprocation theorem gives

$$q_2\frac{\sigma}{\epsilon_0}d + q\frac{\sigma}{\epsilon_0}h = 0.$$

Hence,

$$q_2 = (-q)\frac{h}{d}.$$

Similarly,

$$q_1 = (-q)\frac{d-h}{d}.$$

14. Generally, *Green's Theorem* derives

$$\int_V [G(\mathbf{x},\mathbf{y})\nabla^2 G(\mathbf{x}',\mathbf{y}) - G(\mathbf{x}',\mathbf{y})\nabla^2 G(\mathbf{x},\mathbf{y})]\mathrm{d}\tau = \oint_S \left[G(\mathbf{x},\mathbf{y})\frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right]\mathrm{d}A.$$

Since $\nabla_{\mathrm{y}}^2 G(\mathbf{z},\mathbf{y}) = -4\pi\delta(\mathbf{y}-\mathbf{z})$, the left hand side of the equation can be simplified and then

$$G(\mathbf{x}',\mathbf{x}) - G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi}\oint_S \left[G(\mathbf{x},\mathbf{y})\frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right]\mathrm{d}A.$$

(a) Dirichlet Boundary condition: $G(\mathbf{x},\mathbf{y}) = G(\mathbf{x}',\mathbf{y}) = 0$. Thus,

$$G(\mathbf{x}',\mathbf{x}) - G(\mathbf{x},\mathbf{x}') = 0.$$

(b) Neumann Boundary Condition: $\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} = \frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} = -\frac{4\pi}{S}$. Thus,

$$G(\mathbf{x}',\mathbf{x}) - G(\mathbf{x},\mathbf{x}') = -\frac{1}{S}\oint_S (G(\mathbf{x},\mathbf{y}) - G(\mathbf{x}',\mathbf{y}))\mathrm{d}A.$$

Consequently,

$$G(\mathbf{x},\mathbf{x}') - \frac{1}{S}\oint_S G(\mathbf{x},\mathbf{y})\mathrm{d}A = G(\mathbf{x}',\mathbf{x}) - \frac{1}{S}\oint_S G(\mathbf{x},\mathbf{y})\mathrm{d}A,$$

which shows that $G(\mathbf{x},\mathbf{x}') - F(\mathbf{x})$ is symmetric.

(c) By substituting $G(\mathbf{x},\mathbf{x}')$ by $G(\mathbf{x},\mathbf{x}') - F(\mathbf{x})$ in the E.q. (1.46) on page 39 of the main text, we obtain

$$\langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')(G(\mathbf{x},\mathbf{x}') - F(\mathbf{x}))\mathrm{d}\tau' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}(G(\mathbf{x},\mathbf{x}') - F(\mathbf{x}))\mathrm{d}A'.$$

The surplus term

$$-\frac{1}{4\pi}F(\mathbf{x})\left(\frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}')\mathrm{d}\tau' + \oint_S \nabla\Phi\mathrm{d}\boldsymbol{A}'\right) = 0,$$

according to *Gauss's law*. As a result, the addition of $F(\mathbf{x})$ to the Green function doesn't affect the potential.

15. The total electrostatic energy of the system is

$$W = \frac{\epsilon_0}{2}\int_\infty \boldsymbol{E}^2 \mathrm{d}\tau.$$

The equilibrium of the system requires $\delta W = 0$:

$$\begin{aligned} 0 = \delta W &= \frac{\epsilon_0}{2}\delta \int_\infty \boldsymbol{E}^2 \,\mathrm{d}\tau \\ &= \epsilon_0 \int_\infty \boldsymbol{E}\cdot\delta\boldsymbol{E}\,\mathrm{d}\tau \\ &= -\epsilon_0 \int_\infty \nabla\Phi\cdot\delta\boldsymbol{E}\,\mathrm{d}\tau \\ &= \epsilon_0 \int_\infty \Phi\nabla\cdot\delta\boldsymbol{E}\,\mathrm{d}\tau - \epsilon_0 \oint_\infty V\delta\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{A} \\ &= \int_\infty \Phi\delta\rho\,\mathrm{d}\tau. \end{aligned}$$

Meanwhile, the total charge on the conductor should be constant:

$$\delta Q = \lambda\int_\infty \delta\rho \mathrm{d}\tau.$$

Construct the Lagrangian function $L = W - \lambda Q$, and

$$\delta L = \int_i nfty(\Phi - lambda)\delta\rho\mathrm{d}\tau = \sum_i(\Phi_i - \lambda_i)\delta\rho = 0.$$

Consequently,

$$\Phi_i = \lambda_i,$$

where λ_i are constant.

16. Before introducting the conductor, assume the electric field $\boldsymbol{E}$ is distributed within the volume V, defined as all space excluding the regions occupied by the initial conductors. Similarly, after introducing the conductor, assume the electric field $\boldsymbol{E}'$ is distributed within the volume V'. Then the change of the electrostatic energy of this system:

$$\begin{aligned} \Delta W &= \frac{\epsilon_0}{2}\int_{V'}\boldsymbol{E}'\cdot\boldsymbol{E}'\mathrm{d}\tau - \frac{\epsilon_0}{2}\int_V \boldsymbol{E}\cdot\boldsymbol{E}\mathrm{d}\tau \\ &= -\frac{\epsilon_0}{2}\int_{V-V'}\boldsymbol{E}^2\mathrm{d}\tau - \frac{\epsilon_0}{2}\int_{V'}(\boldsymbol{E}-\boldsymbol{E}')^2\mathrm{d}\tau + \epsilon_0\int_{V'}\boldsymbol{E}'(\boldsymbol{E}'-\boldsymbol{E})\mathrm{d}\tau, \end{aligned}$$

where the last term

$$\begin{aligned} \int_{V'}\boldsymbol{E}'(\boldsymbol{E}'-\boldsymbol{E})\mathrm{d}\tau &= -\int_{V'}\nabla\Phi'\cdot(\boldsymbol{E}'-\boldsymbol{E})\mathrm{d}\tau \\ &= -\int_{V'}\nabla\cdot[\Phi'(\boldsymbol{E}'-\boldsymbol{E})]\mathrm{d}\tau + \int_{V'}\Phi'\nabla\cdot(\boldsymbol{E}'-\boldsymbol{E})\mathrm{d}\tau. \end{aligned}$$

The second term vanishes because there's no electric charge in V', whether it is before the introduction of the new conductor ot after that. Thus,

$$\begin{aligned} \int_{V'}\boldsymbol{E}'(\boldsymbol{E}'-\boldsymbol{E})\mathrm{d}\tau &= -\oint_{\partial V'}\Phi'(\boldsymbol{E}'-\boldsymbol{E})\cdot\mathrm{d}\boldsymbol{A} \\ &= -\oint_\infty \Phi'(\boldsymbol{E}'-\boldsymbol{E})\cdot\mathrm{d}\boldsymbol{A} + \sum_i\oint_{S_i}\Phi'(\boldsymbol{E}'-\boldsymbol{E})\cdot\mathrm{d}\boldsymbol{A} \\ &= \frac{1}{\epsilon_0}\sum_i\Phi_i'\oint_{S_i}(\sigma'-\sigma)\mathrm{d}A \\ &= 0. \end{aligned}$$

Consequently,

$$\Delta W == -\frac{\epsilon_0}{2}\int_{V-V'} \boldsymbol{E}^2 \mathrm{d}\tau - \frac{\epsilon_0}{2}\int_{V'} (\boldsymbol{E}-\boldsymbol{E}')^2 \mathrm{d}\tau < 0,$$

which represents that the electrostatic energy lowers.

17. (a) Suppose the potential of every conducting surface is 0 except S_k. The total electrostatic energy is expressed as

$$W = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}V_iV_j,$$

where $V_i = \delta_{ik}(i = 1, 2, \cdots, N)$. Thus

$$W = \frac{1}{2}C.$$

On the other hand,

$$W = \frac{1}{2}\epsilon_0 \int_V |\nabla\Phi|^2 \mathrm{d}\tau.$$

By combining the two equations,

$$C = \epsilon_0 \int_V |\nabla\Phi|^2 \mathrm{d}\tau.$$

(b) According to Sec. 1.12 in the main text, Φ is the solution which enable the functional

$$I[\psi] = \frac{1}{2}\int |\nabla\psi|^2 \mathrm{d}\tau - \int g\psi \mathrm{d}\tau$$

to reach its extremum (actually minimum). However, because there no charge in V, the source function $g = 0$. Hence,

$$\int_V |\nabla\Phi|^2 \mathrm{d}\tau \leq \int_V |\nabla\psi|^2 \mathrm{d}\tau.$$

Then, the upper bound of the capacitance is

$$C[\psi] = \epsilon_0 \int_V |\nabla\psi|^2 \mathrm{d}\tau.$$

18. (a) Generally, the electrostatic energy can be expressed by

$$W = \frac{1}{2}\sum_{i=1}^{N} \oint_{S_i} \Phi_i(\mathbf{x})\sigma(\mathbf{x})\mathrm{d}a.$$

However all conductors except S_1 held at zero potential, and thus

$$W = \frac{1}{2}\oint_{S_1} \Phi_1(\mathbf{x})\sigma(\mathbf{x})\mathrm{d}a = \frac{1}{8\pi\epsilon_0}\oint_{S_1}\mathrm{d}a \oint_{S_1} \sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\mathrm{d}a'.$$

(b) The electrostatic energy $W = \frac{1}{2}C^{-1}Q^2$, and thus

$$C^{-1} = \frac{1}{Q^2}\frac{1}{4\pi\epsilon_0}\oint_{S_1}\mathrm{d}a\oint_{S_1}\sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\mathrm{d}a' = \frac{\oint_{S_1}\mathrm{d}a\oint_{S_1}\sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\mathrm{d}a'}{4\pi\epsilon_0\left[\oint_{S_1}\sigma(\mathbf{x})\mathrm{d}a\right]^2}.$$

Thomson's theorem states that when the total charge on each conductor is held constant, the electrostatic energy reaches its minimum when all conductors are equipotential. Under this circumstance, C^{-1} reaches its stationary minimum and achieves the lower bound of the capacitance.

19. By substituting $\Psi_1(\rho) = \dfrac{b-\rho}{b-a}$ into the upper bound of the capacitance:

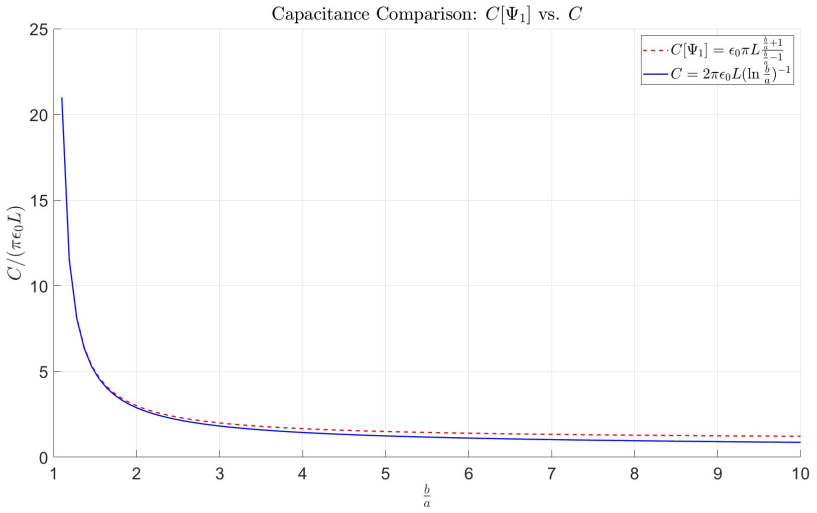
$$C[\Psi_1] = \epsilon_0 \int_V |\nabla\Psi|^2 \mathrm{d}\tau = \epsilon_0 \pi L \frac{\frac{b}{a}+1}{\frac{b}{a}-1}.$$

Meanwhile, the exact solution for the capacitance is given in Problem 1.6(c):

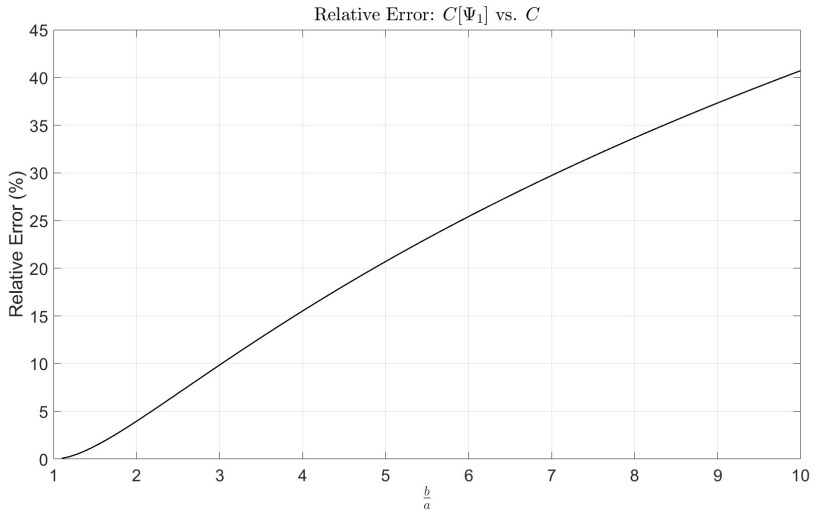
$$C = 2\pi\epsilon_0 L \left(\ln\frac{b}{a}\right)^{-1}.$$

Comparing the approximate results and the exact solution:

$\frac{b}{a}$	$C/(\pi\epsilon_0 L)$	$C[\Psi_1]/(\pi\epsilon_0 L)$	relative error
1.5	4.933	5	1.36%
2	2.885	3	3.99%
3	1.820	2	9.89%



(a)



(b)

20. Assume the potential solution of the configuration S_1 is Φ and the configuration S_1' corresponds to Φ'. Φ satisfied the boundary condition and *Thomson's law* dictates that Φ is uniform inside

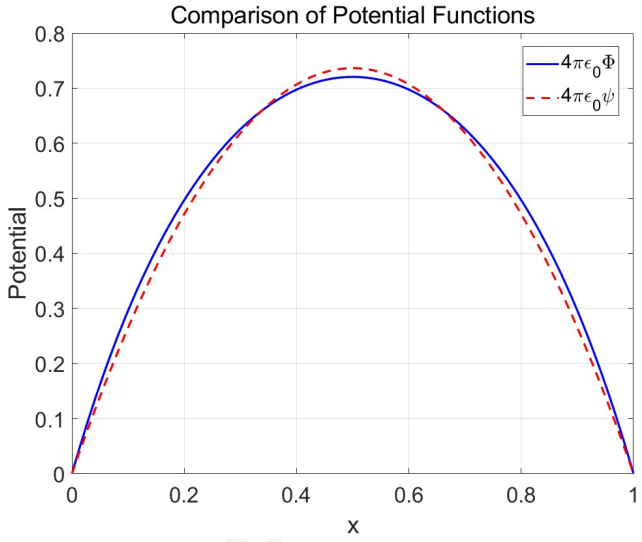
the S_1. Thus Φ can be considered as the trial function of the configuration S_1'. Hence, the variational principle of Problem 1.17 gives

$$C' = \epsilon_0 \int |\nabla\Phi|^2 \mathrm{d}\tau \leq \epsilon_0 \int |\nabla\Phi|^2 \mathrm{d}\tau = C.$$

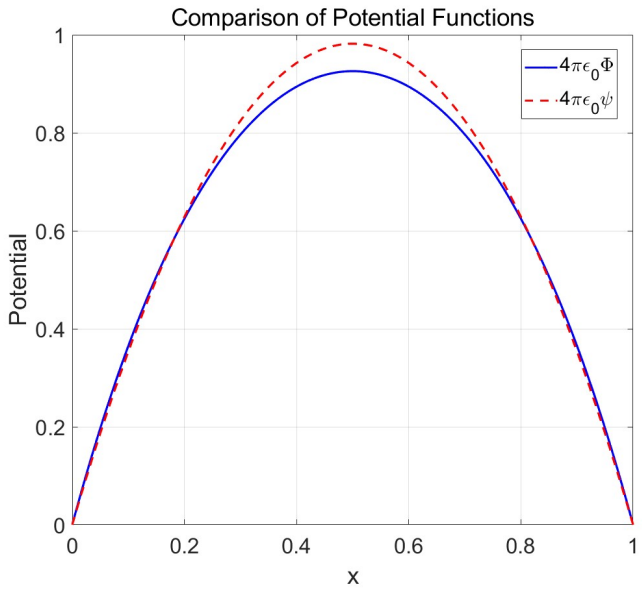
21. Applying the variational principle (1.63), where $g = \frac{1}{\epsilon_0}$:

$$\begin{aligned} I[\psi] &= \frac{1}{2}\int |\nabla|^2 \mathrm{d}S - \int \psi \mathrm{d}S \\ &= A^2 \int_0^1 \int_0^1 (1-2x)^2 y^2 (1-y)^2 \mathrm{d}x\mathrm{d}y - \frac{1}{2}A \int_0^1 \int_0^1 x(1-x)y(1-y)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{90}A^2 - \frac{1}{36\epsilon_0}A. \end{aligned}$$

$I[\psi]$ reaches its minimum when $A = \dfrac{5}{4\epsilon_0}$. The figures of the approximate solution and the exact solution are as followed:



(a) $y = 0.25$



(b) $y = 0.5$

22. (a) The Taylor series expansions of $F(h,0), F(-h,0), F(0,h), F(0,-h)$:

$$F(h,0) = F(0,0) + \frac{\partial F}{\partial x}h + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}h^2 + \frac{1}{6}\frac{\partial^3 F}{\partial x^3}h^3 + \frac{1}{24}\frac{\partial^4 F}{\partial x^4}h^4 + \frac{1}{120}\frac{\partial^5 F}{\partial x^5}h^5 + O(h^6)$$
$$F(-h,0) = F(0,0) - \frac{\partial F}{\partial x}h + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}h^2 - \frac{1}{6}\frac{\partial^3 F}{\partial x^3}h^3 + \frac{1}{24}\frac{\partial^4 F}{\partial x^4}h^4 - \frac{1}{120}\frac{\partial^5 F}{\partial x^5}h^5 + O(h^6)$$
$$F(0,h) = F(0,0) + \frac{\partial F}{\partial y}h + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}h^2 + \frac{1}{6}\frac{\partial^3 F}{\partial y^3}h^3 + \frac{1}{24}\frac{\partial^4 F}{\partial y^4}h^4 + \frac{1}{120}\frac{\partial^5 F}{\partial y^5}h^5 + O(h^6)$$
$$F(0,-h) = F(0,0) - \frac{\partial F}{\partial y}h + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}h^2 - \frac{1}{6}\frac{\partial^3 F}{\partial y^3}h^3 + \frac{1}{24}\frac{\partial^4 F}{\partial y^4}h^4 - \frac{1}{120}\frac{\partial^5 F}{\partial y^5}h^5 + O(h^6).$$

By summing up these equations, we have

$$S_c = 4F(0,0) + h^2\nabla^2 F + \frac{h^4}{12}(F_{xxxx} + F_{yyyy}) + O(h^6).$$

(b) The Taylor series expansions of $F(h,h), F(h,-h), F(-h,h), F(-h,-h)$:

$$F(h,h) = F(0,0) + \sum_{k=1}^{5}\frac{1}{k!}h^k\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^k F + O(h^6)$$
$$F(h,-h) = F(0,0) + \sum_{k=1}^{5}\frac{1}{k!}h^k\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^k F + O(h^6)$$
$$F(-h,h) = F(0,0) + \sum_{k=1}^{5}\frac{1}{k!}h^k\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^k F + O(h^6)$$
$$F(-h,-h) = F(0,0) + \sum_{k=1}^{5}\frac{1}{k!}h^k\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^k F + O(h^6).$$

By summing up these equations, we have

$$\begin{aligned} S_S &= 4F(0,0) + 2h^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F + \frac{1}{6}h^4(F_{xxxx} + F_{yyyy}) + h^4 F_{xxyy} + O(h^6) \\ &= 4F(0,0) + 2h^2\nabla^2 F - \frac{h^4}{3}(F_{xxxx} + F_{yyyy}) + \frac{h^4}{2}\nabla^2(\nabla^2 F) + O(h^6). \end{aligned}$$

23. Omitted.

24. Omitted.

2 Boundary-Value Problems in Electrostatics: I

1. (a) The distribution of the electric field is

$$\boldsymbol{E}(r,\theta) = -\frac{1}{4\pi\epsilon_0}\frac{2qd}{(r^2+d^2)^{3/2}}\hat{\boldsymbol{n}}.$$

Hence, the surface-charge density:

$$\sigma = \epsilon_0 \boldsymbol{E}\cdot\hat{\boldsymbol{n}} = -\frac{1}{4\pi}\frac{4qd}{(r^2+d^2)^{3/2}}.$$

(b) The force between the plane and the charge is equivalent to the force between the charge and its image charge. Hence *Coulomb's law* gives:

$$\boldsymbol{F} = -\frac{q^2}{16\pi\epsilon_0 d^2}\hat{\boldsymbol{n}}.$$

(c) The total force acting on the plane:

$$\boldsymbol{F}' = \int_0^\infty \frac{\sigma^2}{2\epsilon_0}\cdot 2\pi r\mathrm{d}r\ \hat{\boldsymbol{n}} = \frac{q^2}{16\pi\epsilon_0 d^2}\hat{\boldsymbol{n}},$$

which is aligned with (b) and satisfies *Newton's 3rd law*.

(d) The work acting on q to move it from its position to infinity is the necessary (minimum) work when the whole process is quasi-static, neglecting other energy such as the kinetic energy of the charge. Hence the necessary work is

$$W = \int \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s} = \int_0^\infty \frac{1}{4\pi\epsilon_0}\frac{q^2}{4s^2}\mathrm{d}s = \frac{q^2}{16\pi\epsilon_0 d}.$$

(e) The potential between the charge and its image charge is

$$W_{\text{int}} = \frac{q^2}{4\pi\epsilon_0 d}(=2W),$$

which demonstrates that the total interactive energy is allocated equally by the charge and its image charge.

(f) $d = 1\text{Å}, d = e$. Then the necessary work $W = 3.60$ eV.

2. (a) Similar to the case that the point charge is outside the hollow (discussed in Sec. 2.2), the image charge

$$q' = -\frac{a}{d}q$$

is located at $r = b = \frac{a^2}{d}$, outside the hollow. Hence, the potential inside the sphere is

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}+\frac{q'}{r_2}\right),$$

where $r_1 = (r^2+d^2-2rd\cos\theta)^{1/2}$, and $r_2 = (r^2+b^2-2rb\cos\theta)^{1/2}$.

(b) The induced charge density:

$$\sigma = -\epsilon_0\left(\frac{\partial V}{\partial r}\right)_{r=a} = \frac{q}{4\pi a^2}\frac{a(a^2-d^2)}{(a^2+d^2-2ad\cos\theta)^{3/2}}.$$

(c) The force acting on q can be expressed as the force between q and its image charge:

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(b-d)^2}\hat{\boldsymbol{r}} = -\frac{1}{4\pi\epsilon_0 a^2}\left(\frac{d}{a}\right)\frac{q^2}{\left[1-\left(\frac{d}{a}\right)^2\right]^2}\hat{\boldsymbol{r}}.$$

(d) If the sphere is kept at a fixed potential V, then the solution

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}+\frac{q'}{r_2}\right)+V$$

satisfies the Poisson equation and the boundary conditions.

If the sphere has the total charge Q, then the solution

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}+\frac{q'}{r_2}+\frac{Q+q'}{r}\right)$$

satisfies the Poisson equation and the boundary conditions.

3. (a) In the presence of the conductors, one image charge with line charge density λ is located at $(-x_0, y_0)$, and other two image charges with line charge density $-\lambda$ is located at $(x_0, -y_0)$ and $(-x_0, y_0)$. Hence the potential in the first quadrant is

$$\Phi = \frac{\lambda}{4\pi\epsilon_0}\left(\ln\frac{R^2}{r_1^2}+\ln\frac{R^2}{r_2^2}-\ln\frac{R^2}{r_3^2}-\ln\frac{R^2}{r_4^2}\right) = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{r_3^2 r_4^2}{r_1^2 r_2^2},$$

where $r_1^2 = (x-x_0)^2+(y-y_0)^2$, $r_2^2 = (x+x_0)^2+(y+y_0)^2$, $r_3^2 = (x-x_0)^2+(y+y_0)^2$, $r_4^2 = (x+x_0)^2+(y-y_0)^2$.

On the boundary, for instance, $x = 0, y \geq 0$, $r_1 = r_4, y_2 = y_3$, and thus $\Phi = 0$. Meanwhile, the tangential electric field

$$E_\tau = \left(\frac{\partial\Phi}{\partial y}\right)_{x=0} = 0.$$

(b) The generic solution of the charge density is

$$\sigma = -\epsilon_0\left(\frac{\partial\Phi}{\partial y}\right)_{y=0} = -\frac{\lambda}{4\pi}\frac{16x_0y_0x}{[y_0^2+(x-x_0)^2][y_0^2+(x+x_0)^2]}.$$

Hence,

$$\frac{\sigma}{\lambda} = -\frac{4x_0y_0x}{\pi[y_0^2+(x-x_0)^2][y_0^2+(x+x_0)^2]}.$$

The plot is shown in Figure 3.

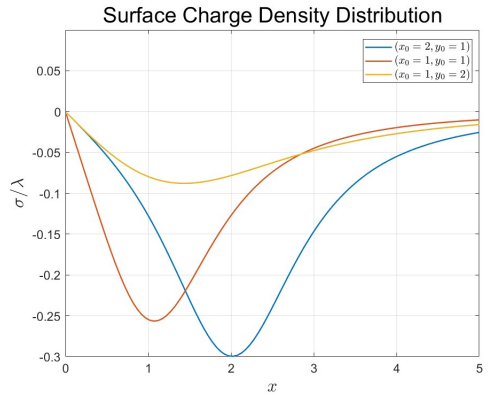


Figure 3: Surface charge density distribution

(c) The total charge on the plane $y = 0, x \geq 0$ is

$$Q_x = \int_0^\infty \sigma \mathrm{d}x = -\frac{\lambda y_0}{\pi} \int_0^\infty \left[\frac{1}{(x-x_0)^2 + y_0^2} - \frac{1}{(x+x_0)^2 + y_0^2} \right] \mathrm{d}x = -\frac{2\lambda}{\pi} \arctan \frac{x_0}{y_0}.$$

(d) When $\rho \gg \rho_0$,

$$\begin{aligned} \frac{r_3^2 r_4^2}{r_1^2 r_2^2} &= \frac{[(x-x_0)^2 + (y+y_0)^2][(x+x_0)^2 + (y-y_0)^2]}{[(x+x_0)^2 + (y+y_0)^2][(x-x_0)^2 + (y-y_0)^2]} \\ &\simeq \frac{[\rho^2 - 2(xx_0 - yy_0)][\rho^2 + 2(xx_0 - yy_0)]}{[\rho^2 - 2(xx_0 + yy_0)][\rho^2 + 2(xx_0 + yy_0)]} \\ &\simeq 1 + \frac{16xyx_0y_0}{\rho^4}. \end{aligned}$$

Hence

$$\Phi = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{r_3^2 r_4^2}{r_1^2 r_2^2} \simeq \frac{4\lambda}{\pi\epsilon_0} \frac{x_0 y_0 xy}{\rho^4}.$$

The potential decreases at the rate of ρ^{-2} when $\rho \gg \rho_0$, which fits the feature of the potential of a quadrupole moment. It is reasonable because when $\rho \gg \rho_0$, the monopole term and the dipole term are both vanished.

4. (a) Suppose the quantity of the charge is q, and thus the image charge of that charge is

$$q_1 = -\frac{R}{d} q\delta(\boldsymbol{r} - b\hat{\boldsymbol{x}}), q_2 = \frac{R+d}{d} q\delta(\boldsymbol{r}).$$

In consequence, the electric field at the position of the charge is

$$E = \frac{1}{4\pi\epsilon_0} \frac{q\left(-\frac{R}{d}q\right)}{\left(d - \frac{R^2}{d}\right)^2} + \frac{1}{4\pi\epsilon_0} \frac{q \cdot \frac{R+d}{d} q}{d^2}.$$

When the charge is attracted by the sphere, $E < 0$, and thus

$$\frac{d}{R} < 1.618.$$

(b) The force of attraction is

$$F = \frac{1}{4\pi\epsilon_0} \frac{q^2(R+d)}{d^3} - \frac{1}{4\pi\epsilon_0} \frac{q^2 Rd}{(R^2 - d^2)^2} = \frac{q^2}{2\pi\epsilon_0 R^2} - \frac{q^2}{16\pi\epsilon_0 a^2},$$

where the first term is much smaller than the second term when $a \to 0$. Hence,

$$F \simeq -\frac{q^2}{16\pi\epsilon_0 a^2}.$$

(c) The approach is similar to (a) and (b). In this case, $\frac{d}{R} < 1.428$, and $F \simeq -\frac{q^2}{16\pi\epsilon_0}$.

5. (a) According to Eq.(2.6) of the main text,

$$F = -\frac{1}{4\pi\epsilon} \frac{q^2}{a^2} \left(\frac{a}{y}\right)^3 \left(1 - \frac{a^2}{y^2}\right)^{-2}.$$

Hence the work:

$$W = -\int_r^\infty F \mathrm{d}y.$$

Let $x = \dfrac{a}{y}$, and

$$W = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\int_0^{a/r}\frac{x}{(x^2-1)^2}\mathrm{d}x = \frac{q^2 a}{8\pi\epsilon_0(r^2-a^2)}.$$

Meantime, the total electric energy can be calculated, according to Sec.1.11 in the main text:

$$W' = \frac{1}{2}\int \rho\Phi \mathrm{d}\tau = \frac{1}{2}Q\Phi_{\text{sphere}} + \frac{1}{2}q\Phi_q,$$

where the sphere is grounded so $\Phi_{\text{sphere}} = 0$, and

$$\Phi_q = -\frac{1}{4\pi\epsilon_0}\frac{a}{r}\frac{q}{r-\frac{a^2}{r}} = -\frac{qa}{4\pi\epsilon_0(r^2-a^2)}.$$

Hence, the total energy is

$$W' = -\frac{q^2 a}{4\pi\epsilon_0(r^2-a^2)} = -W,$$

which is aligned with the former calculation.

(b) According to Eq.(2.9) of the main text,

$$F = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\left(Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right).$$

Hence the work:

$$W = -\int_r^\infty F\mathrm{d}y.$$

Let $x = \dfrac{a}{y}$, and

$$\begin{aligned} W &= -\frac{qQ}{4\pi\epsilon_0}\int_r^\infty \frac{1}{y^2}\mathrm{d}y + \frac{q^2}{4\pi\epsilon_0 a}\int_0^{a/r}\frac{(2-x^2)x^3}{(1-x^2)^2}\mathrm{d}x \\ &= -\frac{qQ}{4\pi\epsilon_0 r} - \frac{q^2}{8\pi\epsilon_0 a}\left(\frac{a^2}{r^2} + \frac{a^2}{a^2-r^2}\right) \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2-a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]. \end{aligned}$$

Meantime, the total electric energy can be calculated, according to Sec.1.11 in the main text:

$$W' = \frac{1}{2}q\Phi + \frac{1}{2}Q\Phi_{\text{sphere}}$$

where

$$\Phi = \frac{1}{4\pi\epsilon_0}\left(\frac{Q+\frac{a}{r}q}{r} + \frac{-\frac{a}{r}q}{r-\frac{a^2}{r}}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{rQ+aq}{r^2} - \frac{aq}{r^2-a^2}\right),$$

and

$$\Phi_{\text{sphere}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2},$$

Hence,

$$W' = \frac{1}{4\pi\epsilon_0}\left[\frac{qQ}{r^2} + \frac{q^2 a}{2r^2} - \frac{q^2 a}{2(r^2-a^2)}\right] = -W,$$

which is aligned with the former calculation.

Note: Here, we calculate the total energy by considering the sphere as a whole, instead of using the method of image charge, because the energy of the original charge and the two image charges isn't always same as the energy between the original charge and the sphere, though it can simplified the case.

6. (a) The iterative formula are easy to verify by applying Eq.(2.4) of the main text repetitively.

(b) Here is part of the key MATLAB codes of the calculation:

```
clear all;

%% Parameters
epsilon0 = 8.854187817e-12;    % vacuum permittivity
ra = 1.0;       % Radius of A
rb = 1.0;       % Radius of B
Qa = 1e-6;      % Electric charge of A
Qb = -1e-6;     % Electric charge of B
d = 100.0;        % Distance between the centers of A and B
maxIter = 500;  % Iteration times
tol = 1e-6;     % Threshold of the potential difference

%% Iteration
[qa, qb, xa, xb] = deal(zeros(maxIter,1));

% initial condition (j=1)
qa(1) = 100; xa(1) = 0; da = d;
qb(1) = -100; xb(1) = 0; db = d;

% Iteration calculation of the image charges
for j = 2:maxIter
        qa(j) = -ra*qb(j-1)/db;
        xa(j) = ra^2/db;
        da = d - xa(j);

        qb(j) = -rb*qa(j-1)/da;
        xb(j) = rb^2/da;
        db = d - xb(j);
end

% Normalization of the image charges
qa = Qa * qa / sum(qa);
qb = Qb * qb / sum(qb);

%% Potential Calculation and checking the Equipotential
% Cheking points
checkPoints = struct(...
        'sphereA_pole', [ra,0,0],...   % Pole of A (near B)
        'sphereA_equator', [0,ra,0],...    % Equator of A
        'sphereB_pole', [d-rb,0,0],...  % Pole of B (near A)
```

```
        'sphereB_equator', [d, rb,0]);      % Equator of B

% Potential calculation function
V = @(r_test, charges, positions) sum(charges./(4*pi*epsilon0*
    vecnorm(r_test - positions,2,2)));

% Potential calculation for each cheking points
VA_pole = V(checkPoints.sphereA_pole, qb, [d-xb, zeros(size(xb))
    , zeros(size(xb))]) + ...
        V(checkPoints.sphereA_pole, qa, [xa, zeros(size(xa)),
            zeros(size(xa))]);

VA_equator = V(checkPoints.sphereA_equator, qb, [d-xb, zeros(
    size(xb)), zeros(size(xb))]) + ...
        V(checkPoints.sphereA_equator, qa, [xa, zeros(size(xa)),
             zeros(size(xa))]);

VB_pole = V(checkPoints.sphereB_pole, qa, [xa, zeros(size(xa)),
    zeros(size(xa))]) + ...
        V(checkPoints.sphereB_pole, qb, [d-xb, zeros(size(xb)),
            zeros(size(xb))]);

VB_equator = V(checkPoints.sphereB_equator, qa, [xa, zeros(size(
    xa)), zeros(size(xa))]) + ...
        V(checkPoints.sphereB_equator, qb, [d-xb, zeros(size(xb)
            ), zeros(size(xb))]);

%% Force Calculation
F = 0;
for i = 1:maxIter
        for j = 1:maxIter
                r_ij = abs( (d - xb(j)) - xa(i) ); % Distance
                    between the charges
                F = F + (qa(i)*qb(j))/(4*pi*epsilon0*r_ij^2);
        end
end
```

7. (a) The Green function satisfies $\nabla^2 G = -\dfrac{\delta(\boldsymbol{x}-\boldsymbol{x}')}{\epsilon_0}$, and the boundary condition: $G(z=0)=0$. Hence the Green function can be derived from the method of image charge:

$$G(\boldsymbol{x},\boldsymbol{x}') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right].$$

In the cylindrical coordinate system (r, θ, z), Green function can be expressed as

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}} - \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2}}\right].$$

(b) The formal solution of the Poisson Equation $\nabla^2\Phi = -\dfrac{\rho}{\epsilon_0}$ is given in Sec.1.10 in the main text:

$$\Phi(\boldsymbol{r}) = \int_V \rho(\boldsymbol{r})G(\boldsymbol{r}, \boldsymbol{r}')\mathrm{d}\tau' - \epsilon_0 \oint_S \Phi(\boldsymbol{r}')\frac{\partial G}{\partial n'}\mathrm{d}S'.$$

In this case, $\rho = 0$ in the upper half-space, and $\Phi = 0$ on the plane $z = 0$ out of the circle. Hence,

$$\Phi(\boldsymbol{r}) = -\epsilon_0 V \int_{S'} \frac{\partial G}{\partial z'}\mathrm{d}S',$$

where $S' = \{(r, \theta, z)|r < a, 0 \le \theta \le 2\pi, z = 0\}$. According to (a),

$$\left(\frac{\partial G}{\partial z'}\right)_{z'=0} = -\frac{1}{4\pi\epsilon_0}\frac{2z}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}$$

Hence,

$$\Phi(\boldsymbol{r}) = \frac{V}{2\pi}\int_S \frac{z}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}\mathrm{d}S'. \qquad (*)$$

(c) When $\rho = 0$, the integral in (b) is

$$\Phi(\boldsymbol{r}) = \frac{zV}{2\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_0^a \frac{\rho'}{[\rho'^2+z^2]^{3/2}}\mathrm{d}\rho' = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right).$$

(d) Because the solution is azimuthally symmetric, it is reasonable to consider the case of $\phi = 0$. Then, by expanding the Eq.(*) in a power series in $(\rho^2+z^2)^{-1}$, we have:

$$\begin{aligned}
\Phi(\boldsymbol{r}) &= \frac{Vz}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_S \frac{\rho'\mathrm{d}\rho'\mathrm{d}\phi'}{\left[1+\frac{\rho'^2-2\rho\rho'\cos\phi'}{\rho^2+z^2}\right]^{3/2}} \\
&= \frac{V}{2\pi}\frac{z}{(\rho^2+z^2)^{3/2}}\int_0^{2\pi}\mathrm{d}\phi'\int_0^a\left[1-\frac{3}{2}\frac{\rho'^2-2\rho\rho'\cos\phi'}{\rho^2+z^2}+\frac{15}{8}\left(\frac{\rho'^2-2\rho\rho'\cos\phi'}{\rho^2+z^2}\right)^2+\cdots\right] \\
&= \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2}+\cdots\right].
\end{aligned}$$

8. Suppose the charge density at $\left(\dfrac{R}{2}, 0\right)$ is λ and the charge density at $\left(-\dfrac{R}{2}, 0\right)$ is $-\lambda$. Suppose the potential at the origin is 0.

(a) The potential at any point (x, y) can be expressed as

$$\Phi = \Phi_+ + \Phi_- = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{R/2}{r_+} - \frac{\lambda}{2\pi\epsilon_0}\ln\frac{R/2}{r_-} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_-}{r_+}.$$

Let $\Phi = V$,

$$\frac{r_-}{r_+} = e^\alpha,$$

where $\alpha \equiv \dfrac{2\pi\epsilon_0 V}{\lambda}$. This property is aligned with a well-know circle, *Apollonius* circle. By simplifying

$$\frac{\left(x+\frac{R}{2}\right)^2+y^2}{\left(x-\frac{R}{2}\right)^2+y^2} = e^{2\alpha},$$

we can get the function of the circle:

$$\left(x-\frac{1}{2}\cosh\alpha R\right)^2+y^2=\left(\frac{R}{2\sinh\alpha}\right)^2.$$

Hence, the coodinates of the axis of the cylinder is $\left(\frac{1}{2}\coth\alpha R,0\right)$, and the radius is $\left|\frac{R}{2\sinh\alpha}\right|$.

(b) Choose the two conductors overlap the equipotential surfaces with potential V_1, V_2 in (a). Suppose the radius of the conductor with positive potential $V_1(>0)$ is a, and the radius of the conductor with negative potential $V_2(<0)$ is b. Thus $\alpha\equiv\frac{2\pi\epsilon_0 V_1}{\lambda}>0$, and $\beta\equiv\frac{2\pi\epsilon_0 V_2}{\lambda}<0$. Meanwhile,

$$a=\frac{R}{2\sinh\alpha},\quad b=-\frac{R}{2\sinh\beta},\quad d=\frac{R}{2}(\coth\alpha-\coth\beta).$$

Hence,

$$\sinh\alpha=\frac{R}{2a},\quad \cosh\alpha=\frac{\sqrt{R^2+4a^2}}{2a},\quad \sinh\beta=-\frac{R}{2b},\quad \cosh\beta=\frac{\sqrt{R^2+4b^2}}{2b}.$$

By substituting these formula into the former equation related to d, we have

$$d=\frac{\sqrt{R^2+4a^2}+\sqrt{R^2+4b^2}}{2}.$$

Consequently,

$$R^2=\left(\frac{a^2+d^2-b^2}{d}\right)^2-4a^2.$$

Additionally,

$$\frac{2d}{R}=\coth\alpha-\coth\beta=\frac{\sinh(\beta-\alpha)}{\sinh\alpha\sinh\beta}=\frac{4ab}{R^2}\sinh(\alpha-\beta).$$

Hence,

$$\cosh(\alpha-\beta)=\sqrt{\sinh^2(\alpha-\beta)+1}=\sqrt{\frac{d^2R^2+4a^2b^2}{2ab}}=\frac{d^2-a^2-b^2}{2ab}.$$

Then, the capacitance per unit length is

$$C=\frac{\lambda}{V_1-V_2}=\frac{2\pi\epsilon_0}{\alpha-\beta}=\frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)}.$$

(c) $\frac{1}{\sqrt{x^2-1}}$ can be expressed as a series expansion:

$$\frac{1}{\sqrt{x^2-1}}=\frac{1}{x\sqrt{1-1/x^2}}=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}x^{-2n-1}.$$

Integral both sides of the equation[1]:

$$\cosh^{-1}x=\ln 2x-\sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n)}x^{-2n}.$$

[1] In principle, it is necessary to proof that the series above is convergent uniformly. But in fact, it is easy to determine that this series is uniformly convergent around $\infty(|x|>1)$, by taking the transformation $t=1/x$.

Thus,

$$\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right) \simeq \ln\frac{d^2-a^2-b^2}{ab} \simeq \ln\frac{d^2}{ab}.$$

Consequently,

$$C \simeq \pi\epsilon_0 \left(\ln\frac{d}{\sqrt{ab}}\right)^{-1},$$

which is align with the answer in Problem 1.7.

To determine the next nonvanishing order correction, we need to review the series expansion of $\cosh^{-1} x$:

$$\begin{aligned}\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right) &\simeq \ln\frac{d^2-a^2-b^2}{ab} - \frac{1}{4}\left(\frac{2ab}{d^2-a^2-b^2}\right)^2 \\ &= \ln\frac{d^2}{ab} + \ln\left(1-\frac{a^2}{d^2}-\frac{b^2}{d^2}\right) - \frac{a^2}{d^2}\frac{b^2}{d^2}\left(1-\frac{a^2}{d^2}-\frac{b^2}{d^2}\right)^{-2} \\ &\simeq \ln\frac{d^2}{ab} - \left(\frac{a}{d}\right)^2 - \left(\frac{b}{d}\right)^2 + \mathcal{O}\left[\left(\sqrt{\frac{a^2}{d^2}+\frac{b^2}{d^2}}\right)^4\right].\end{aligned}$$

Hence,

$$C \simeq \frac{2\pi\epsilon_0}{\ln\frac{d^2}{ab} - \left[\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2\right]} \simeq \frac{2\pi\epsilon_0}{\ln\frac{d^2}{ab}} + \frac{2\pi\epsilon_0}{\left(\ln\frac{d^2}{ab}\right)^2}\left[\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2\right].$$

(d) For the case of $d < |b-a|$, the result is similar:

$$C = \frac{\lambda}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\alpha-\beta} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a^2+b^2-d^2}{2ab}\right)}.$$

For the concentric cylinder, i.e. $d = 0$, the capacitance is

$$C = \frac{\lambda}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\alpha-\beta} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a^2+b^2}{2ab}\right)} = \frac{2\pi\epsilon_0}{|\ln(b/a)|},$$

which is aligned with what we have already known.

9. (a) According to the result in Sec. 2.5 in the main text, the induced charge density is given by Eq.(2.15):

$$\sigma = 3\epsilon_0 E_0 \cos\theta.$$

Then the force on unit area is

$$\boldsymbol{f} = \frac{\sigma^2}{2\epsilon_0}\hat{\boldsymbol{r}} = \frac{9\epsilon_0 E_0^2\cos^2\theta}{2}\hat{\boldsymbol{r}}.$$

Hence the total force on one hemisphere is

$$F = \int f\cos\theta \mathrm{d}S = \int_0^{\pi/2} f\cdot 2\pi a^2 \sin\theta\cos\theta \mathrm{d}\theta = \frac{9}{4}\pi\epsilon_0 E_0^2 a^2.$$

(b) When the total charge on the shell is Q, the surface charge density should be modified by a uniformly distribution term, i.e.

$$\sigma = 3\epsilon_0 E_0\cos\theta + \frac{Q}{4\pi a^2}.$$

Then the force on unit area is

$$\boldsymbol{f} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{r}} = \left(\frac{9}{2}\epsilon_0 E_0^2\cos^2\theta + \frac{Q^2}{32\epsilon_0\pi^2 a^4} + \frac{3QE_0\cos\theta}{4\pi a^2}\right)\hat{\boldsymbol{r}}.$$

Hence the total force on one hemisphere is

$$\begin{aligned} F &= \frac{9}{4}\pi\epsilon_0 E_0^2 a^2 + \int_0^{\pi/2}\frac{Q^2}{32\epsilon_0\pi^2 a^4}2\pi a^2\sin\theta\cos\theta\mathrm{d}\theta + \int_0^{\pi/2}\frac{3QE_0\cos\theta}{4\pi a^2}2\pi a^2\sin\theta\cos\theta\mathrm{d}\theta \\ &= \frac{9\pi\epsilon_0 E_0^2 a^2}{4} + \frac{Q^2}{32\pi\epsilon_0 a^2} + \frac{1}{2}QE_0. \end{aligned}$$

CAUTION: not all the terms above correspond to the separated force! In fact, all terms are related to the separated force except the third term, which corresponds to the response of the initial total charge to the external field.

Consequently, the exact separated force should be

$$F = \frac{9\pi\epsilon_0 E_0^2 a^2}{4} + \frac{Q^2}{32\pi\epsilon_0 a^2}.$$

10. (a) The charge density on the boss is given by Eq.(2.15) in Sec.2.5 of the main text:

$$\sigma = 3\epsilon_0 E_0\cos\theta.$$

The potential at an arbitrary point is given by Eq.(2.14) in the main text:

$$\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta.$$

Thus the charge density on the plane is

$$\sigma = -\nabla\Phi\Big|_{\theta=\frac{\pi}{2}}\cdot\hat{\boldsymbol{\theta}} = -\epsilon_0 E_0\left(1 - \frac{a^3}{r^3}\right).$$

(b) The total charge on the boss is

$$Q_{\mathrm{tot}} = \int\sigma\mathrm{d}S = \int_0^{\pi/2} 3\epsilon_0 E_0\cos\theta\cdot 2\pi a^2\sin\theta\mathrm{d}\theta = 3\pi\epsilon_0 E_0 a^2.$$

(c) When the other plate is replaced by a point charge q, the remained plate will induce an image charge $-q$ first, and then each of these two charge will respectively induce a image charge $q' = -aq/d$ inside the boss. Consequently, the potential at an arbitrary point is

$$\Phi = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2} - \frac{aq/d}{r_3} + \frac{aq/d}{r_4}\right),$$

where

$$\begin{aligned} r_1 &= \sqrt{r^2 + d^2 - 2rd\cos\theta}, \\ r_2 &= \sqrt{r^2 + d^2 + 2rd\cos\theta}, \\ r_3 &= \sqrt{r^2 + \left(\frac{a^2}{d}\right)^2 - 2r\frac{a^2}{d}\cos\theta}, \\ r_4 &= \sqrt{r^2 + \left(\frac{a^2}{d}\right)^2 + 2r\frac{a^2}{d}\cos\theta}. \end{aligned}$$

Hence, the charge density on the boss is

$$\sigma' = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a} = \frac{q}{4\pi}\frac{a^2-d^2}{a}\left[\frac{1}{(a^2+d^2-2ad\cos\theta)^{3/2}} - \frac{1}{(d^2+a^2+2ad\cos\theta)^{3/2}}\right],$$

and the total charge is

$$q' = \int_0^{\pi/2} \sigma' \cdot 2\pi a^2 \sin\theta \mathrm{d}\theta = -q\left(1 - \frac{d^2-a^2}{d\sqrt{a^2+d^2}}\right).$$

11. (a) Suppose the magnitude of the image charge is $\tau' = k\tau$, and it is at d away from the central axis. Suppose the voltage of the conductor is V_0, and then potential at an arbitrary point on the surface can be expressed as:

$$\frac{\tau}{2\pi\epsilon_0}\ln\frac{r_0}{r} + \frac{\tau}{2\pi\epsilon_0}\ln\frac{r_0}{r'} = V_0,$$

where $r^2 = R^2 + b^2 - 2Rb\cos\theta$, and $r'^2 = d^2 + b^2 - 2bd\cos\theta$. Hence

$$(b^2 + R^2 - 2Rb\cos\theta)(b^2 + d^2 - 2bd\cos\theta)^k = \text{const.}$$

The terms in both brackets is a linear function with respect to $\cos\theta$, so to let the product be constant, k is supposed to be -1, i.e.

$$b^2 + R^2 - 2Rb\cos\theta = \lambda(b^2 + d^2 - 2bd\cos\theta)$$

for arbitrary θ, where λ is a constant coefficient. Thus,

$$b^2 + R^2 = \lambda(b^2+d^2), \quad 2Rb = 2\lambda bd.$$

Consequently, the non-trivial solution of d is

$$d = \frac{b^2}{R}.$$

In conclusion,

$$\tau' = -\tau, \quad d = \frac{b^2}{R}.$$

NOTE: Of course, it is easy to examine that this solution is aligned with the answer of PROBLEM 2-8.(a).

(b) The potential can be expressed as the superposition of the potential of the original charge and the image charge:

$$\Phi = \frac{\tau}{2\pi\epsilon_0}\ln\frac{r_0}{r_1} - \frac{\tau}{2\pi\epsilon_0}\ln\frac{r_0}{r_2} = \frac{\tau}{2\pi\epsilon_0}\ln\frac{r_2}{r_1} = \frac{\tau}{4\pi\epsilon_0}\ln\frac{r^2+d^2-2rd\cos\theta}{r^2+R^2-2rR\cos\theta},$$

where $d = \dfrac{b^2}{R}$. To get the asymptotic form of the solution, the log term can be expressed as

$$\ln\frac{1+\frac{d^2}{r^2}-2\frac{d}{r}\cos\theta}{1+\frac{R^2}{r^2}-2\frac{R}{r}\cos\theta} \simeq \ln\frac{1-2\frac{d}{r}\cos\theta}{1-2\frac{R}{r}\cos\theta} = \sum_{k=1}^{\infty}\frac{(R^k-d^k)2^k\cos^k\theta}{k}\left(\frac{1}{r^k}\right).$$

Consequently,

$$\Phi \sim \frac{\tau}{4\pi\epsilon_0}\sum_{k=1}^{\infty}\frac{(R^k-d^k)2^k\cos^k\theta}{k}\left(\frac{1}{r^k}\right).$$

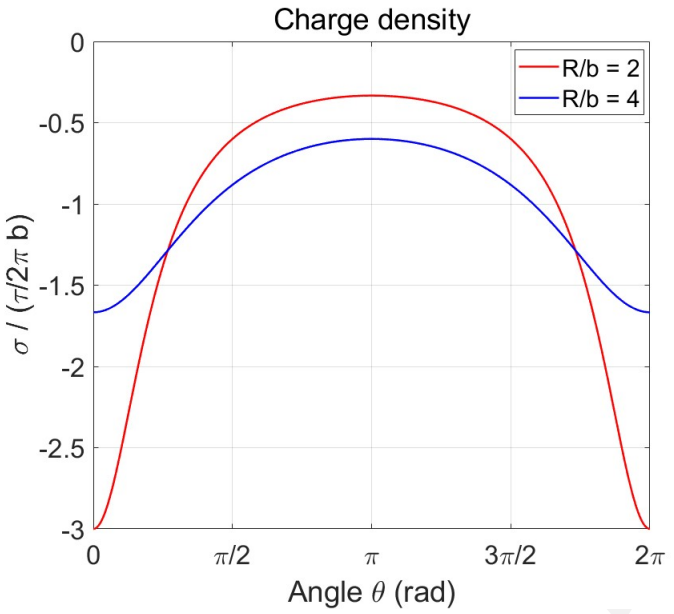


Figure 4: Charge density distribution

(c) The induced charge density is

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=b} = -\frac{\tau}{2\pi b}\frac{R^2 - b^2}{R^2 + b^2 - 2bR\cos\theta}.$$

The graph of the charge density distribution is shown as Fig.4.

(d) The electric field at the position of the line charge is

$$\boldsymbol{E} = -\nabla\Phi = -\frac{\partial \Phi}{\partial r}\Big|_{(r,\theta)=(R,0)}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\Big|_{(r,\theta)=(R,0)}\hat{\boldsymbol{\theta}} = -\frac{\tau}{2\pi\epsilon_0(R-d)}\hat{\boldsymbol{r}}.$$

Hence the force per unit length is

$$\boldsymbol{f} = \tau\boldsymbol{E} = -\frac{\tau^2 R}{2\pi\epsilon_0(R^2 - b^2)}\hat{\boldsymbol{r}}$$

12. Eq.(2.71) gives:

$$\Phi = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \cos(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n). \tag{2.71}$$

While for the case of the potential inside the cylinder, Φ must be finite when $\rho \to 0$. Hence the formal solution of the potential should be written as

$$\Phi = A_0 + \sum_{n=1}^{\infty} A_n \rho^n \cos n\phi + \sum_{n=1}^{\infty} B_n \rho^n \sin n\phi.$$

Hence

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\mathrm{d}\phi',$$

$$A_n = \frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b,\phi')\cos n\phi'\mathrm{d}\phi',$$

$$B_n = \frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b,\phi')\sin n\phi'\mathrm{d}\phi'.$$

By substituting these coefficients into the series solution and summing them up, we can obtain

$$\Phi = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\left[1+\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}\cos n(\phi'-\phi)\right]\mathrm{d}\phi',$$

where the terms in the bracket can be transformed into

$$\begin{aligned}1+2\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}\cos n(\phi'-\phi) &= \Re\left[-1+2\sum_{n=0}^{\infty}\left(\frac{\rho}{b}e^{\mathrm{i}(\phi'-\phi)}\right)^n\right]\\ &= \Re\left[-1+\frac{2}{1-\frac{\rho}{b}e^{\mathrm{i}(\phi'-\phi)}}\right]\\ &= \Re\left(\frac{b^2-\rho^2+2\mathrm{i}b\rho\sin(\phi'-\phi)}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\right)\\ &= \frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}.\end{aligned}$$

Finally, the potential inside the cylinder is expressed in the form of Poisson integral:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\mathrm{d}\phi'.$$

To calculate the potential in the region out of the cylinder, it is necessary to ensure Φ is finite when $\rho\to\infty$, i.e. $b_0(n=1,2,\cdots)$ in Eq.(2.71).

13. (a) The potential on the surface of the cylinder is

$$\Phi(b,\phi) = \begin{cases} V_1, & -\frac{\pi}{2}<\phi<\frac{\pi}{2},\\ V_2, & \frac{\pi}{2}<\phi<\frac{3\pi}{2}.\end{cases}$$

By substituting it into the solution of Problem 2.12, we can obtain

$$\Phi = \frac{V_1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\mathrm{d}\phi' + \frac{V_2}{2\pi}\int_{\pi/2}^{3\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\mathrm{d}\phi'.$$

Our calculation will start from the indefinite integral:

$$\begin{aligned}\int\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\mathrm{d}\phi' &\xlongequal{t=\tan\frac{\phi'-\phi}{2}} 2(b^2-\rho^2)\int\frac{\mathrm{d}t}{(b-\rho)^2+(b+\rho^2)t^2}\\ &\xlongequal{s=\frac{b+\rho}{b-\rho}t} 2\int\frac{\mathrm{d}s}{1+s^2} = 2\arctan s\\ &= 2\arctan\left(\frac{b+\rho}{b-\rho}\tan\frac{\phi'-\phi}{2}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\mathrm{d}\phi' &= 2\arctan\left[\frac{b+\rho}{b-\rho}\tan\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right]+2\arctan\left[\frac{b+\rho}{b-\rho}\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]\\ &= 2\arctan\frac{\frac{b+\rho}{b-\rho}\frac{1-\tan\phi/2}{1+\tan\phi/2}+\frac{b+\rho}{b-\rho}\frac{1+\tan\phi/2}{1-\tan\phi/2}}{1-\left(\frac{b+\rho}{b-\rho}\right)^2}\\ &= \pi+2\arctan\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right).\end{aligned}$$

And similarly,

$$\int_{\pi/2}^{3\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\mathrm{d}\phi' = \pi-2\arctan\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right).$$

Consequently,

$$\Phi(\rho,\phi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\arctan\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right).$$

(b) The surface charge density can be derived from the potential:

$$\sigma=-\epsilon_0\frac{\partial\Phi}{\partial\rho}\Big|_{\rho=b}=\epsilon_0\frac{V_1-V_2}{\pi b\cos\phi}.$$

14. (a) When $\rho\to 0$, Φ is supposed to be finite. Thus the coefficients of $\ln\rho$ and ρ^{-n} are 0. Due to the distribution on the surface

$$\Phi(b,\phi)=\begin{cases}V, & -\pi<\phi<-\dfrac{\pi}{2}\text{ or }0<\phi<\dfrac{\pi}{2},\\ -V, & -\dfrac{\pi}{2}<\phi<0\text{ or }\dfrac{\pi}{2}<\phi<\pi,\end{cases}$$

the coefficients of $\sin n\phi$ are 0. Hence the series solution is

$$\Phi=A_0+\sum_{n=1}^{\infty}A_n\rho^n\sin n\phi.$$

The coefficients are:

$$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(b,\phi')\mathrm{d}\phi'=0,$$

$$A_n=\frac{V}{\pi b^n}\int_{-\pi}^{\pi}\Phi(b,\phi')\sin n\phi'\mathrm{d}\phi'=\frac{2V}{\pi nb^n}\left(1+\cos n\pi-2\cos\frac{1}{2}n\pi\right)=\begin{cases}\dfrac{8V}{\pi nb^n}, & n=4k+2,\\ 0, & \text{others.}\end{cases}.$$

Consequently,

$$\Phi=\frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin(4n+2)\phi}{2n+1}.$$

(b) Further consolidating the previous findings:

$$\begin{aligned}\Phi&=\frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin(4n+2)\phi}{2n+1}=\frac{8V}{\pi}\int_0^{\phi}\sum_{0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\cos(4n+2)\phi'\mathrm{d}\phi'\\&=\frac{8V}{\pi}\int_0^{\phi}\Re\left[\frac{(\rho/b)^2e^{2\mathrm{i}\phi}}{1-(\rho^4/b^4)e^{4\mathrm{i}\phi}}\right]\mathrm{d}\phi'=\frac{8V}{\pi}\int_0^{\phi}\frac{\rho^2b^2(b^4-\rho^4)\cos2\phi'}{\rho^8+b^8-2\rho^4b^4\cos4\phi'}\mathrm{d}\phi'\\&=\frac{2V}{\pi}\arctan\left(\frac{2\rho^2b^2}{b^4-\rho^4}\sin2\phi\right).\end{aligned}$$

(c) The field lines and the equipotentials can be drawn by COMSOL (Fig.5).

15. (a) According to the definition of the Green function, $G(x,y;x',y')$ is supposed to satisfy:

$$\nabla'^2G(x,y;x',y')=-4\pi\delta(x-x')\delta(y-y').$$

For the given expansion,

$$\begin{aligned}\nabla'^2G(x,y;x',y')&=\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}\right)G\\&=2\sum_{n=1}^{\infty}\left(\frac{\partial^2}{\partial y'^2}-n^2\pi^2\right)g(y,y')\sin(n\pi x)\sin(n\pi x')\\&=-8\pi\delta(y-y')\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x'),\end{aligned}$$

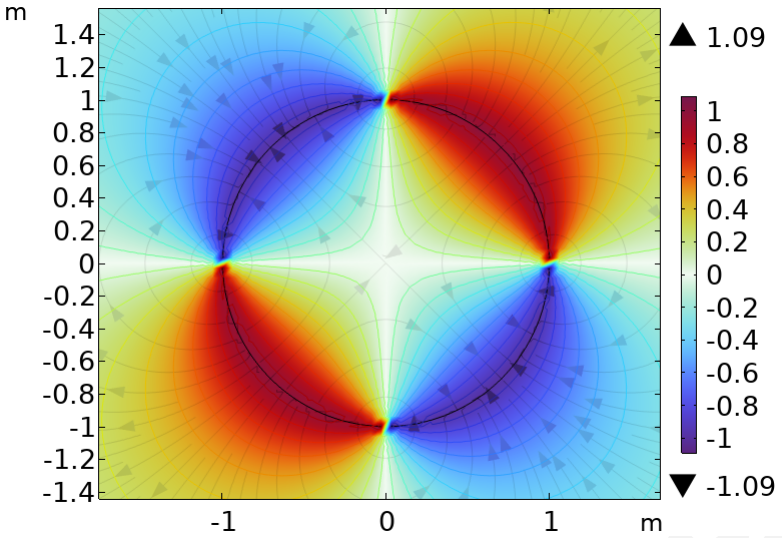


Figure 5: Field lines and equipotentials

where

$$\begin{aligned} 2\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x') &= \sum_{n=1}^{\infty}[\cos(n\pi(x-x')) - \cos[n\pi(x+x')]] \\ &= \Re\sum_{n=1}^{\infty}\left[e^{in\pi(x-x')} - e^{in\pi(x+x')}\right] \\ &= \begin{cases} \frac{1}{2} - \frac{1}{2} = 0, & x \neq x', \\ \infty, & x = x'. \end{cases} \\ &= \delta(x-x'). \end{aligned}$$

Meanwhile, $G(x, y; x', y') = 0$ on the boundary.

Consequently, $G(x, y; x', y')$ is the Green function appropriating for Dirichlet boundary conditions.

(b) When $y \neq y'$, $g_n(y, y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y, y') = 0.$$

Hence,

$$g_n(y, y') = \begin{cases} A\sinh(n\pi y') + B\cosh(n\pi y'), & y > y', \\ A'\sinh(n\pi y') + B'\cosh(n\pi y'), & y < y', \end{cases}$$

where $g_n(y, 0) = 0$ dictates that

$$B = 0,$$

and $g_n(y, 1) = 0$ dictate that

$$A' = -B'\frac{\cosh(n\pi)}{\sinh(n\pi)}.$$

Hence,

$$g_n(y, y') = \begin{cases} A\sinh(n\pi y'), & y > y', \\ \dfrac{B'}{\sinh(n\pi)}\sinh(n\pi(1-y')), & y < y'. \end{cases}$$

Because Green function has its symmetry, it can be written in a uniform form when $y \neq y'$:

$$g_n(y, y') = C \sinh(n\pi y_<) \sinh[n\pi(1 - y_>)],$$

where C is the coefficient we need to determine.

Next, we go back to the equation

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y, y') = -4\pi\delta(y - y'),$$

and integrate both sides from $y - \epsilon$ to $y + \epsilon$, where ϵ is an infinitesimal quantity. Consequently, we obtain

$$\frac{\mathrm{d}g_n(y, y')}{\mathrm{d}y'}\Big|_{y+\epsilon} - \frac{\mathrm{d}g_n(y, y')}{\mathrm{d}y'}\Big|_{y-\epsilon} = -4\pi,$$

where both terms can be easily calculated (take the limit $\epsilon \to 0$):

$$\frac{\mathrm{d}g_n(y, y')}{\mathrm{d}y'}\Big|_{y+\epsilon} = -n\pi C \sinh(n\pi y) \cosh[n\pi(1 - y)], \quad y = y_+,$$

$$\frac{\mathrm{d}g_n(y, y')}{\mathrm{d}y'}\Big|_{y-\epsilon} = n\pi C \cosh(n\pi y') \sinh[n\pi(1 - y)], \quad y = y_-.$$

Finally, C is determined as

$$C = \frac{4}{n \sinh(n\pi)},$$

and we obtain

$$G(x, y; x', y') = 8 \sum_{n=1}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(n\pi x) \sin(n\pi x') \sinh(n\pi y_<) \sinh[n\pi(1 - y_>)].$$

16. Using the Green function of Problem 2.15, the potential can be written as

$$\begin{aligned}
\Phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0} \int_D \rho(\boldsymbol{r}') G(\boldsymbol{r}, \boldsymbol{r}') \tau' \\
&= \frac{1}{4\pi\epsilon_0} \int_D 8 \sum_{n=1}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(n\pi x) \sin(n\pi x') \sinh(n\pi y_<) \sinh[n\pi(1 - y_>)] \mathrm{d}x' \mathrm{d}y' \\
&= \frac{2}{\pi\epsilon_0} \int_{n=0}^{\infty} \frac{\sin n\pi x}{n \sinh(n\pi)} \left[\int_0^1 \sin(n\pi x') \mathrm{d}x'\right] \left[\int_0^1 \sinh(n\pi y_<) \sinh[n\pi(1 - y_>)] \mathrm{d}y'\right],
\end{aligned}$$

where the first integral is

$$\frac{1}{n\pi}[1 - (-1)^n] = \begin{cases} \dfrac{2}{n\pi}, & n = 2k + 1, \\ 0, & n = 2k. \end{cases},$$

and the second integral is

$$\begin{aligned}
&\int_0^y \sinh(n\pi y') \sinh[n\pi(1 - y)] \mathrm{d}y' + \int_y^1 \sinh(n\pi y) \sinh[n\pi(1 - y')] \mathrm{d}y' \\
=&\frac{1}{n\pi} \sinh n\pi - \frac{1}{n\pi} [\sinh[n\pi(1 - y)] + \sinh(n\pi y)] \\
=&\frac{2}{n\pi} \sin \frac{n\pi}{2} \left[\cosh \frac{n\pi}{2} - \cos \frac{n\pi(2y - 1)}{2}\right].
\end{aligned}$$

Hence,

$$\Phi(\boldsymbol{r}) = \frac{4}{\pi^3\epsilon_0} \sum_{n=0}^{\infty} \frac{\sin[(2m + 1)\pi x]}{(2m + 1)^3} \left\{1 - \frac{\cosh[(2m + 1)\pi(y - \frac{1}{2})]}{\cosh[(2m + 1)\pi/2]}\right\}.$$

17. (a) The free-space Green function for three-dimensional case is

$$G(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{R},$$

where $R=\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$. Hence, for two-dimensional case, the Green function should be

$$G(x,y;x',y')=\int_{-Z}^{Z}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\mathrm{d}z'=\ln(z-z'+\sqrt{r^2+(z-z')^2})\Big|_{-Z}^{Z},$$

where $r^2=(x-x')^2+(y-y')^2$. As Z is very large,

$$\ln(Z-z+\sqrt{r^2+(Z-z')^2})\simeq\ln 2+\ln(Z-z),$$

$$\begin{aligned}\ln(-Z-z+\sqrt{r^2+(Z+z)^2})&=\ln(Z+z)+\ln\left(-1+\sqrt{1+\frac{r^2}{(Z+z)^2}}\right)\\&\simeq-\ln(Z+z)-\ln 2+\ln r^2.\end{aligned}$$

Consequently,

$$G(x,y;x',y')=\ln(Z^2-z^2)+2\ln 2-\ln r^2,$$

where the first two terms are inessential constants, so that finally

$$G(x,y;x',y')=-\ln r^2=-\ln[(x-x')^2+(y-y')^2]=-\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')].$$

(b) Green function in polar coordinates is supposed to satisfy:

$$\nabla'^2G=-\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi'),$$

where $\delta(\phi-\phi')$ can be expressed in Fourier series expansion (the system has a period of 2π):

$$\delta(\phi-\phi')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{\mathrm{i}m(\phi-\phi')}.$$

By separation of variables, Green function can be expressed as

$$G=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{\mathrm{i}m(\phi-\phi')}g_m(\rho,\rho'),$$

and then substitute it into the Poisson equation, $g_m(\rho,\rho')$ satisfied:

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m(\rho,\rho')}{\partial\rho'}\right)-\frac{m^2}{\rho'^2}g_m(\rho,\rho')=-\frac{4\pi}{\rho'}\delta(\rho-\rho').\qquad(*)$$

(c) • $m>0$: When $\rho\neq\rho'$, the solution can be expressed as

$$g_m(\rho,\rho')=A\rho'^m+B\rho'^{-m}.$$

However, the boundary condition dictates that g_m is finite when $\rho'\to 0$ and $\rho\to\infty$, and thus

$$g_m(\rho,\rho')=\begin{cases}A\rho'^m, & \rho>\rho',\\ B\rho'^{-m}, & \rho<\rho'.\end{cases}$$

Because Green function is symmetric with regard to ρ and ρ', it can be written in a uniform form:

$$g_m(\rho,\rho') = C\left(\frac{\rho_<}{\rho_>}\right)^m,$$

where C is a constant. Then, go back to Eq.(*):

$$\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m(\rho,\rho')}{\partial\rho'}\right) - \frac{m^2}{\rho'}g_m(\rho,\rho') = -4\pi\frac{\rho'\delta(\rho-\rho')}{\rho}$$

and integral with regard to ρ' from $\rho-\epsilon$ to $\rho'+\epsilon$. We can consequently derive

$$\rho'\frac{\mathrm{d}g_m}{\mathrm{d}\rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\mathrm{d}g_m}{\mathrm{d}\rho'}\Big|_{\rho-\epsilon} = -4\pi,$$

where

$$\rho'\frac{\mathrm{d}g_m}{\mathrm{d}\rho'}\Big|_{\rho+\epsilon} = -\rho'\frac{\mathrm{d}g_m}{\mathrm{d}\rho'}\Big|_{\rho-\epsilon} = -mC, \quad \epsilon\to 0.$$

Hence, $C = \dfrac{2\pi}{m}$.

- $m=0$: When $\rho\neq\rho'$, the solution can be expressed as

$$g_0(\rho,\rho') = A + B\ln\rho'.$$

Meanwhile, the boundary condition dictates that

$$g_0 \text{ is finite}, \quad \rho'\to 0,$$

$$g_0 \sim \ln\rho', \quad \rho'\to\infty,$$

where the second condition can be derived from (a). With the consistent condition at $\rho=\rho'$,

$$A = C + D\ln\rho.$$

Hence,

$$g_0(\rho,\rho') = \begin{cases} A, & \rho'<\rho, \\ A + D\ln\rho' - D\ln\rho, & \rho<\rho'. \end{cases}$$

When $A = D\ln\rho$, Green function will satisfy the symmetry with regard to ρ and ρ', and thus g_0 can be written in the uniform form

$$g_0(\rho,\rho') = D\ln\rho_>.$$

By doing the similar integral as the case of $m\neq 0$, we can obtain a same result

$$\rho'\frac{\mathrm{d}g_m}{\mathrm{d}\rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\mathrm{d}g_m}{\mathrm{d}\rho'}\Big|_{\rho-\epsilon} = -4\pi,$$

where

$$\rho'\frac{\mathrm{d}g_m}{\mathrm{d}\rho'}\Big|_{\rho+\epsilon} = D, \quad \rho'\frac{\mathrm{d}g_m}{\mathrm{d}\rho'}\Big|_{\rho-\epsilon} = 0.$$

Hence $C = -4\pi$.

Finally, by substituting $g_m(m=0,1,\cdots)$ into the result of (b), the Green function can be written as

$$\begin{aligned} G(\rho,\phi;\rho',\phi') &= -\ln(\rho_>^2) + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\left(e^{\mathrm{i}m(\phi-\phi')} + e^{-\mathrm{i}m(\phi-\phi')}\right) \\ &= -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cdot\cos[m(\phi-\phi')]. \end{aligned}$$

18. (a) The process to calculate g_m in the series expansion is similar to Problem 2.17.(c).

- $m = 0$: When $\rho \neq \rho'$, the general solution can be expressed as

$$g_0(\rho, \rho') = \begin{cases} A_0 + B_0 \ln \rho', & \rho' < \rho, \\ A_0' + B_0' \ln \rho', & \rho' > \rho, \end{cases}$$

where $B_0 = 0$ because g_0 is finite when $\rho' \to 0$, and

$$g_0(\rho, b) = A_0' + B_0' \ln b = 0.$$

Hence,

$$g_0(\rho, \rho') = \begin{cases} A_0 + B_0 \ln \rho', & \rho' < \rho, \\ B_0' \ln \dfrac{\rho'}{b}, & \rho' > \rho. \end{cases}$$

The uniform form of g_0 is

$$g_0 = C_0 \ln \frac{\rho_>}{b},$$

where

$$C_0 = -4\pi$$

can be derived from the similar method in Problem 2.17.(c).

- $m \neq 0$: When $\rho \neq \rho'$, the general solution can be expressed as

$$g_m(\rho, \rho') = \begin{cases} A_m \rho'^m + B_m \rho'^{-m}, & \rho' < \rho, \\ A_m' \rho'^m + B_m' \rho'^{-m}, & \rho' > \rho, \end{cases}$$

where $B_m = 0$ and

$$A_m' b^m + B_m' b_m'^{-m} = 0$$

because of boundary conditions. Hence the uniform form of g_m is

$$g_m = C_m \rho_<^m \left(\rho_>^m - \frac{b^{2m}}{\rho_>^m} \right),$$

where

$$C_m = -\frac{2\pi}{m b^{2m}}.$$

Collectively, the Green function in this case can be expressed as the series expansion:

$$G(\rho, \phi; \rho', \phi') = -\ln \left(\frac{\rho_>}{b} \right)^2 - 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{b} \right)^m \left[\left(\frac{\rho_>}{b} \right)^m - \left(\frac{b}{\rho_>} \right)^m \right] \cos[m(\phi - \phi')].$$

To get the closed form, without loss of generality, let $\rho_< = \rho'$, $\rho_> = \rho$, and then

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho\rho'}{b^2} \right)^m \cos[m(\phi - \phi')] &= \Re \left[\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho'\rho}{b^2} e^{\mathrm{i}(\phi - \phi')} \right)^m \right] \\ &= -\Re \left[\ln \left(1 - \frac{\rho\rho'}{b^2} e^{\mathrm{i}(\phi - \phi')} \right) \right] \\ &= -\ln \left| 1 - \frac{\rho\rho'}{b^2} e^{\mathrm{i}(\phi - \phi')} \right|, \end{aligned}$$

and similarly,

$$\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho'}{\rho}\right)^m\cos[m(\phi-\phi')]=-\ln\left|1-\frac{\rho'}{\rho}e^{\mathrm{i}(\phi-\phi')}\right|.$$

By simplifying the formula, the closed form can be expressed as

$$G(\rho,\phi;\rho',\phi')=\ln\frac{b^4+\rho^2\rho'^2-2b^2\rho\rho'\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi'))}$$

or

$$G(\rho,\phi;\rho',\phi')=\ln\frac{(b^2-\rho^2)(b^2-\rho'^2)+b^2|\boldsymbol{\rho}-\boldsymbol{\rho}'|}{b^2|\boldsymbol{\rho}-\boldsymbol{\rho}'|}.$$

(b) There's no charge distributed inside the cylinder, so the potential is

$$\Phi(\rho,\phi)=-\frac{1}{4\pi}\oint_S\Phi(b,\phi')\frac{\partial G}{\partial\rho'}\mathrm{d}S,$$

where

$$\frac{\partial G}{\partial\rho'}=2\cdot\frac{\rho'(\rho^4-b^4)+\rho(b^2-\rho^2)(b^2+\rho^2)\cos(\phi-\phi')}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')][b^4+\rho^2\rho'^2-2b^2\rho\rho'\cos(\phi-\phi')]}.$$

Hence,

$$\Phi(\rho,\phi)=-\frac{b}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{\rho'(\rho^4-b^4)+\rho(b^2-\rho^2)(b^2+\rho^2)\cos(\phi-\phi')}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')][b^4+\rho^2\rho'^2-2b^2\rho\rho'\cos(\phi-\phi')]}\mathrm{d}\phi'.$$

(c) As for the exterior problem, the Fourier form can be derived by swapping $\rho_</b$ and $b/\rho_>$, and the closed form can be derived by changing $b^2-\rho^2$ and $b^2-\rho'^2$ into ρ^2-b^2 and ρ'^2-b^2 respectively (in fact, the result doesn't change formally.)

19. Omitted. (The process is akin to that in Problem 2.17 and Problem 2.18.)

20. (a) The potential can be calculated by

$$\Phi(\boldsymbol{r})=\frac{1}{4\pi\epsilon_0}\int_S\sigma(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}S.$$

- As for the term of $m=0$,

$$\begin{aligned}V_0&=\frac{\lambda}{a}\sum_{n=0}^{3}(-)^{n+1}\int_0^{2\pi}\delta\left(\phi'-\frac{n\pi}{2}\right)\mathrm{d}\phi'\int_0^{\infty}\ln\rho_>^2\delta(\rho'-a)\rho'\mathrm{d}\rho'\\&=\frac{\lambda}{a}\sum_{n=0}^{3}(-)^{n+1}\left[\int_0^{\rho}\rho'\ln\rho'^2\delta(\rho'-a)\mathrm{d}\rho'+\int_{\rho}^{\infty}\rho'\ln\rho^2\delta(\rho'-a)\mathrm{d}\rho'\right]\\&=\lambda\sum_{n=0}^{3}(-)^{n+1}\ln\rho_<^2=0,\end{aligned}$$

  where $\rho_<=\min\{\rho,a\}$.

- As for the term of $m\neq0$,

$$\begin{aligned}V_m&=\frac{2\lambda}{am}\sum_{n=0}^{3}(-)^n\int_0^{2\pi}\cos[m(\phi-\phi')]\delta\left(\phi'-\frac{n\pi}{2}\right)\mathrm{d}\phi'\int_0^{\infty}\left(\frac{\rho_<}{\rho_>}\right)^m\delta(\rho'-a)\rho'\mathrm{d}\rho'\\&=\frac{2\lambda}{am}\sum_{n=0}^{3}(-)^n\cos\left[m\left(\phi-\frac{n\pi}{2}\right)\right]\left[\int_0^{\rho}\left(\frac{\rho'}{\rho}\right)^m\rho'\delta(\rho'-a)\mathrm{d}\rho'+\int_{\rho}^{\infty}\left(\frac{\rho}{\rho'}\right)^m\delta(\rho'-a)\mathrm{d}\rho'\right]\\&=\frac{2\lambda}{am}\sum_{n=0}^{3}(-)^n\cos\left[m\left(\phi-\frac{n\pi}{2}\right)\right]\left(\frac{\rho_<}{\rho_>}\right)^m,\end{aligned}$$

where $\rho <= \min\{\rho, a\}, \rho_> = \max\{\rho, a\}$, and

$$\sum_{n=0}^{3}(-)^n \cos\left[m\left(\phi - \frac{n\pi}{2}\right)\right] = \begin{cases} 0, & m = 4k, 4k+1, 4k+3, \\ 4\cos(4k+2)\phi, & m = 4k+2. \end{cases}$$

Collectively, the electric potential is

$$\Phi(\phi, \rho) = \frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{\rho_<}{\rho_>}\right)^{4k+2}\cos[(4k+2)\phi].$$

(b) The solution of (a) can be transformed into the closed form:

$$\Phi(\phi, \rho) = \Re\left[\frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{\rho_<}{\rho_>}e^{\mathrm{i}\phi}\right)^{4k+2}\right].$$

Let $S(x) = \sum_{k=0}^{\infty}\frac{1}{2k+1}x^{4k+2}$,

$$\frac{\mathrm{d}S(x)}{\mathrm{d}x} = 2\sum_{k=0}^{\infty}x^{4k+1} = \frac{2x}{1-x^4}.$$

Hence,

$$S(x) = \int_0^x \frac{2t}{1-t^4}\mathrm{d}t = \frac{1}{2}\ln\frac{1+x^2}{1-x^2}.$$

Consequently,

$$\Phi(\rho, \phi) = \Re\left[\frac{2\lambda}{4\pi\epsilon_0}\ln\frac{1+\frac{\rho_<^2}{\rho_>^2}e^{2\mathrm{i}\phi}}{1-\frac{\rho_<^2}{\rho_>^2}e^{2\mathrm{i}\phi}}\right] = \begin{cases} \Re[w(z)], & \rho < a, \\ \Re[w(z^*)] = \Re[w^*(z)], & \rho > a. \end{cases}$$

Notes that the coordinate of four linear charges on the complex plane are $z = a, z = \mathrm{i}a, z = -a, z = -\mathrm{i}a$. Hence

$$\Re[w(z)] = \frac{2\lambda}{4\pi\epsilon_0}\ln\frac{|z-\mathrm{i}a||z+\mathrm{i}a|}{|z-a||z+a|} = \frac{2\lambda}{4\pi\epsilon_0}\ln\frac{r_3 r_4}{r_1 r_2},$$

where r_1, r_2, r_3, r_4 are defined in Problem 2.3.(a), and the result are also aligned with what we obtain in Problem 2.3.

(c) Based on the conclusion in (b), the potential can be expressed in Cartesian coordinates:

$$\begin{aligned}\Phi(x, y) = \Re[w(z)] =& \frac{2\lambda}{4\pi\epsilon_0}\ln\frac{|x+\mathrm{i}(y-a)||x+\mathrm{i}(y+a)|}{|(x-a)+\mathrm{i}y||(x+a)+\mathrm{i}y|} \\ =& \frac{\lambda}{4\pi\epsilon_0}\ln\frac{[x^2+(y-a)^2][x^2+(y+a)^2]}{[(x-a)^2+y^2][(x+a)^2+y^2]}.\end{aligned}$$

Hence,

$$E_x = -\frac{\partial\Phi}{\partial x} = -\frac{2\lambda x a^2}{\pi\epsilon_0}\frac{2(y^4-x^4)+[(x^2+y^2+a^2)^2-a^2x^2-a^2y^2]}{[x^2+(y-a)^2][x^2+(y+a)^2][(x-a)^2+y^2][(x+a)^2+y^2]}.$$

When $\rho \to 0$,

$$E_x \simeq -\frac{2\lambda}{\pi\epsilon_0 a^2}x.$$

The y components can also be derived in the similar way:

$$E_y \simeq \frac{2\lambda}{\pi\epsilon_0 a^2}y.$$

On the other hand, by calculating the series expansion form, we can obtain the contribution to E_x (also E_y) of multipole terms. The potential of $k = 0$ term, which corresponds to quadrupole term, is

$$\Phi_0 = \frac{\lambda}{\pi\epsilon_0 a^2}\rho^2 \cos 2\phi = \frac{\lambda}{\pi\epsilon_0 a^2}(x^2 - y^2).$$

Hence the contribution to E_x is

$$E_x^{(0)}\Big|_{y=0} = -\frac{\partial \Phi_0}{\partial x}\Big|_{y=0} = -\frac{2\lambda}{\pi\epsilon_0 a^2}x,$$

which indicates that quadrupole term is the major contributor to E_x when $\rho \to 0$.

As for $k = 1$ term,

$$\Phi_1 = \frac{\lambda}{3\pi\epsilon_0}\frac{\rho^6}{a^6}\cos 6\phi = \frac{\lambda}{3\pi\epsilon_0}\frac{13x^6 - 13y^6 + 3x^2y^4 - 3x^4y^2}{a^6},$$

and

$$E_x^{(1)}\Big|_{y=0} = -\frac{\partial \Phi_1}{\partial x}\Big|_{y=0} = -\frac{2\lambda}{\pi\epsilon_0 a^2}x \cdot 13\left(\frac{x}{a}\right)^4.$$

Hence,

$$\frac{E_x^{(1)}}{E_x^{(0)}} = 13\left(\frac{x}{a}\right)^4 = \mathcal{O}\left[\left(\frac{x}{a}\right)^4\right]$$

21. $\Phi(\rho, \phi)$, satisfying Laplace's Equation in $\mathcal{R}$, is harmonic function of two variables, so it can be considered as the real part of a analytic function $\Phi(z)$. Now, let's use Cauchy's Theorem to derive Poisson's integral solution in Problem 2.12. Hence the boundary $\mathcal{C}$ is a circle centered at the origin with radius b. For any $z \in \mathcal{R}$, $\dfrac{b^2}{z^*} \notin \mathcal{R}$. Hence,

$$\Phi(z) = \frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}} \Phi(z')\left(\frac{1}{z' - z} - \frac{1}{z' - b^2/z^*}\right)\mathrm{d}z',$$

where $z = \rho e^{\mathrm{i}\phi}$, $z' = be^{\mathrm{i}\phi'}$. Thus,

$$\mathrm{d}z' = \mathrm{i}be^{\mathrm{i}\phi'}\mathrm{d}\phi',$$

$$\frac{1}{z' - z} = \frac{1}{be^{\mathrm{i}\phi'} - \rho e^{\mathrm{i}\phi}} = \frac{be^{-\mathrm{i}\phi'} - \rho e^{-\mathrm{i}\phi}}{b^2 + \rho^2 - 2b\rho\cos(\phi - \phi')},$$

$$\frac{1}{z' - b^2/z} = \frac{1}{be^{\mathrm{i}\phi'} - b^2/\rho e^{\mathrm{i}\phi}} = \frac{\rho}{b}\frac{\rho e^{-\mathrm{i}\phi'} - be^{-\phi}}{b^2 + \rho^2 - 2b\rho\cos(\phi - \phi')}.$$

Consequently,

$$\Phi(z) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(z)\frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\phi - \phi')}\mathrm{d}\phi',$$

and

$$\Phi(\rho, \phi) = \Re[\Phi(z)] = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b, \phi')\frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\phi - \phi')}\mathrm{d}\phi',$$

which is aligned with the Poisson integral in Problem 2.12.

22. (a) The Green function is

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} + \frac{-a/r'}{|\boldsymbol{r} - (a^2/r'^2)\boldsymbol{r}'|} = \frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right)^{1/2}},$$

where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$.

The interior potential can be expressed as

$$\Phi(\boldsymbol{r}) = -\frac{1}{4\pi}\oint_S \Phi(\boldsymbol{r}')\frac{\partial G}{\partial n'}\mathrm{d}S',$$

where

$$\frac{\partial G}{\partial n'}\Big|_{r'=a} = -\frac{\partial G}{\partial r'}\Big|_{r'=a} = \left[\frac{r'-r\cos\gamma}{(r^2+r'^2-2rr'\cos\gamma)^{3/2}} - \frac{r'r^2/a^2 - r\cos\gamma}{\left(\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma\right)^{3/2}}\right]_{r'=a}$$
$$= -\frac{1}{a}\frac{r^2-a^2}{(r^2+a^2-2ar\cos\gamma)^{3/2}}.$$

To calculate the potential along z axis, r should be substituted by z and γ should be substituted by zenith angle θ. Consequently,

$$\Phi(z) = \frac{aV}{2}\int_0^{\pi/2}\frac{z^2-a^2}{(z^2+a^2-2az\cos\theta)^{3/2}}\sin\mathrm{d}\theta - \frac{aV}{2}\int_{\pi/2}^{\pi}\frac{z^2-a^2}{(z^2+a^2-2az\cos\theta)^{3/2}}\sin\theta\mathrm{d}\theta,$$

Finally, the solution can be derived:

$$\Phi(z) = V\frac{a}{z}\left[1-\frac{(a^2-z^2)}{a\sqrt{a^2+z^2}}\right].$$

By expanding the result in powers of z, we obtained:

$$\Phi(z) = V\frac{a}{z}\left[1-\frac{1-z^2/a^2}{\sqrt{1+z^2/a^2}}\right] = V\frac{a}{z}\left[1-\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{1}{2}\frac{z^2}{a^2}+\frac{3}{8}\frac{z^4}{a^4}-\frac{5}{16}\frac{z^6}{a^6}+\cdots\right)\right],$$

$$\Phi(z) = V\left[\frac{3}{2}\left(\frac{z}{a}\right)-\frac{7}{8}\left(\frac{z}{a}\right)^3+\frac{11}{16}\left(\frac{z}{a}\right)^5-\cdots\right].$$

(b) The radial electric field can be easily derived by calculating the negative derivative of $\Phi(z)$ with respect to z. When $z\to 0$,

$$E_r(z) \simeq -\frac{V}{a}\left[\left(3+\frac{a^2}{z^2}\right)\left(1-\frac{3}{2}\frac{z^2}{a^2}\right)-\frac{a^2}{z^2}\right] = -\frac{V}{a}\left(\frac{3}{2}-\frac{9}{2}\frac{z^2}{a^2}\right) \simeq -\frac{3V}{2a},$$

which implies that the field is well behaved at the origin. The radial field at the north pole inside is derived from the latter formula by letting $z=a$:

$$E_r(a) = -\frac{V}{a}\left(\frac{4}{2\sqrt{2}}-1\right) = -\frac{(\sqrt{2}-1)V}{a}.$$

And the field at the north pole outside is derived from the former formula by letting $z=a$:

$$E_r(a) = \frac{4Va^2}{2\sqrt{2}a^3} = \frac{\sqrt{2}V}{a}.$$

(c) Electric field lines can be visualized by COMSOL, shown in Fig.6.

And the figure of the distribution of radial electric field is shown as Fig.7.

23. (a) According to the method of separation of variables, the potential can be expressed as

$$\Phi(x,y,z) = X(x)Y(y)Z(z),$$

where

$$X(x) = A\sin\alpha_m x + B\cos\alpha_m x,$$

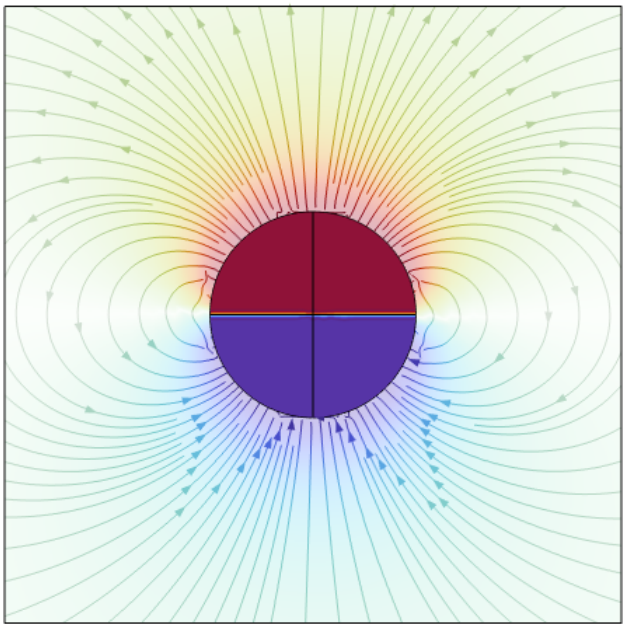


Figure 6: A stretch of the electric field lines (the color reflects the distribution of potential)

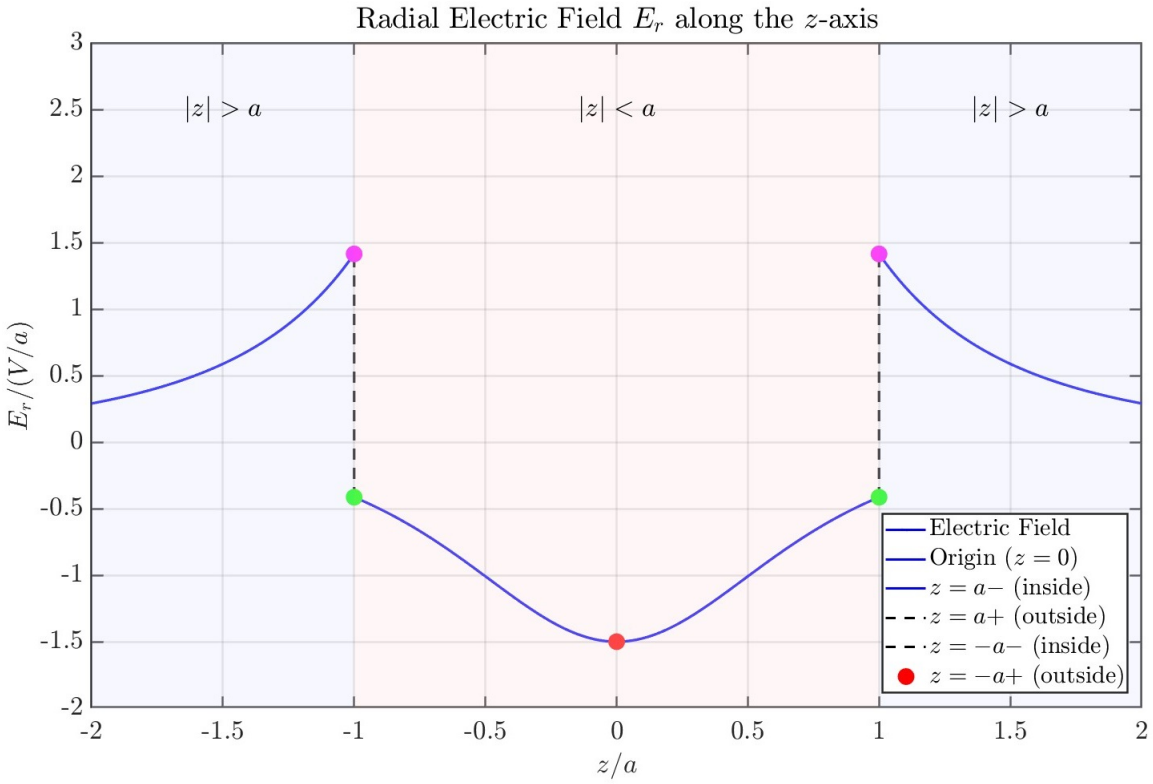


Figure 7: Radial Electric Field E_r along the z-axis

$$Y(y) = C\sin\beta_n y + D\cos\beta_n y,$$
$$Z(z) = E\sinh\sqrt{\alpha_m^2+\beta_n^2}z + F\cosh\sqrt{\alpha_m^2+\beta_n^2}z.$$

$\Phi(x=0)=\Phi(x=a)=0$ dictates that

$$B=0, \quad \alpha_m = \frac{m\pi}{a}, m\in\mathbb{N}^*.$$

$\Phi(y=0)=\Phi(y=a)$ dictates that

$$D=0, \quad \beta_n = \frac{n\pi}{a}, n\in\mathbb{N}^*.$$

$\Phi(z=0)=\Phi(z=a)=V$ dictates that the system is symmetric with respect to the plane $z=\dfrac{a}{2}$, and thus

$$Z(z)\propto\cosh\left[\sqrt{m^2+n^2}\left(\frac{\pi z}{a}-\frac{\pi}{2}\right)\right].$$

Hence,

$$\Phi(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\cosh\left[\sqrt{m^2+n^2}\left(\frac{\pi z}{a}-\frac{\pi}{2}\right)\right],$$

where A_{mn} is the Fourier coefficient satisfies

$$\frac{V}{\cosh[\sqrt{m^2+n^2}\pi/2]}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right).$$

Apply a double Fourier sine series expansion to V, the coefficients can be determined by

$$\begin{aligned}A_{mn}&=\frac{4V}{a^2\cosh[\sqrt{m^2+n^2}\pi/2]}\int_0^a\int_0^a\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\mathrm{d}x\mathrm{d}y\\&=\frac{4V}{mn\pi^2\cosh[\sqrt{m^2+n^2}\pi/2]}\left[1-(-1)^m\right]\left[1-(-1)^n\right].\end{aligned}$$

(b) The potential at the center of the cubic is

$$\Phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$

The terms don't vanish when m and n are odd simultaneously. The contribution of each term are as follows:

m	n	Φ
1	1	$0.3475V$
1	3	$-0.0075V$
3	1	$-0.0075V$
1	5	$0.0002V$
3	3	$0.0005V$
5	1	$0.0002V$
1	7	$-7\times10^{-6}V$
3	5	$-2\times10^{-5}V$
5	3	$-2\times10^{-5}V$
7	1	$-7\times10^{-6}V$

Hence, $\Phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) \simeq 0.333V$.

(c) The charge density on the surface $z = a$ is

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}\Big|_{z=a} = \epsilon_0 \frac{\partial \Phi}{\partial z}\Big|_{z=a}.$$

Thus,

$$\sigma(x,y) = \frac{4V}{a\pi} \tanh\left[\sqrt{m^2+n^2}\frac{\pi}{2}\right] \sum_{m,n=1}^{\infty} \frac{[1-(-1)^m][1-(-1)^n]\sqrt{m^2+n^2}}{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right).$$

24. Suppose

$$\delta(\phi-\phi') = \sum_{m,n=1}^{\infty} A_{mn} \sin\frac{m\pi\phi}{\beta} \sin\frac{m\pi\phi'}{\beta}.$$

Hence

$$\begin{aligned} A_{mn} &= \frac{4}{\beta^2}\int_0^\beta \int_0^\beta \delta(\phi-\phi') \sin\frac{m\pi\phi}{\beta} \sin\frac{m\pi\phi'}{\beta} \mathrm{d}\phi \mathrm{d}\phi' \\ &= \frac{4}{\beta}\int_0^\beta \sin^2\frac{m\pi\phi}{\beta} \mathrm{d}\phi = \frac{2}{\beta}. \end{aligned}$$

25. (a) Based on the conclusion of 2.25, the Green Function can be expressed as

$$G = \frac{2}{\beta}\sum_{m=1}^{\infty} \sin\frac{m\pi\phi}{\beta} \sin\frac{m\pi\phi'}{\beta} g_m(\rho,\rho').$$

G satisfies

$$\nabla'^2 G = -4\pi\delta(\rho-\rho')\delta(\phi-\phi').$$

Hence, $g_m(\rho,\rho')$ satisfies

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{1}{\rho'^2}\frac{m^2\pi^2}{\beta^2} g_m = -4\pi(\rho-\rho').$$

- When $m = 0$, the general solution is

$$g_0 = A_0 + B_0 \ln\rho'.$$

However, g_0 is supposed to be 0 when $\rho' \to 0$ and $\rho' \to \infty$. Thus,

$$A_0 = B_0 = 0.$$

- When $m \neq 0$, the general solution is

$$g_m = A_m \rho'^{m\pi/\beta} + B_m \rho'^{-m\pi/\beta}.$$

Suppose

$$g_m = \begin{cases} A_m \rho'^{m\pi/\beta} + B_m \rho'^{-m\pi/\beta}, & \rho' < \rho, \\ A'_m \rho'^{m\pi/\beta} + B'_m \rho'^{-m\pi/\beta}, & \rho' > \rho. \end{cases}$$

$g_m \to 0$ when $\rho' \to 0$ or ∞. Hence $B_m = A'_m$, and the unified solution is

$$g(\rho,\rho') = C_m \rho_<^{m\pi/\beta} \rho_>^{-m\pi/\beta},$$

where the coefficient

$$C_m = \frac{2\beta}{m}$$

can be defined by using the method akin to 2.17.(c).

To conclusion,

$$G = 4\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}\sin\frac{m\pi\phi}{\beta}\sin\frac{m\pi\phi'}{\beta}.$$

(b) Green function can be transformed as followed:

$$\begin{aligned}
G &= 4\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}\sin\frac{m\pi\phi}{\beta}\sin\frac{m\pi\phi'}{\beta}\\
&= 2\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}\cos\frac{m\pi}{\beta}(\phi-\phi') - 2\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}\cos\frac{m\pi}{\beta}(\phi+\phi')\\
&= 2\Re\left[\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<^{\pi/\beta}}{\rho_>^{\pi/\beta}}e^{\mathrm{i}(\phi-\phi')\pi/\beta}\right)^m - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<^{\pi/\beta}}{\rho_>^{\pi/\beta}}e^{\mathrm{i}(\phi+\phi')\pi/\beta}\right)^m\right]\\
&= 2\Re\left[-\ln\left(1-\frac{\rho_<^{\pi/\beta}}{\rho_>^{\pi/\beta}}e^{\mathrm{i}(\phi-\phi')\pi/\beta}\right) + \ln\left(1-\frac{\rho_<^{\pi/\beta}}{\rho_>^{\pi/\beta}}e^{\mathrm{i}(\phi+\phi')\pi/\beta}\right)\right]\\
&= \ln\frac{\left|\rho_>^{\pi/\beta}-\rho_<^{\pi/\beta}e^{\mathrm{i}(\phi+\phi')\pi/\beta}\right|^2}{\left|\rho_>^{\pi/\beta}-\rho_<^{\pi/\beta}e^{\mathrm{i}(\phi-\phi')\pi/\beta}\right|^2}\\
&= \ln\left(\frac{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2\rho_>^{\pi/\beta}\rho_<^{\pi/\beta}\cos(\phi+\phi')\pi/\beta}{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2\rho_>^{\pi/\beta}\rho_<^{\pi/\beta}\cos(\phi-\phi')\pi/\beta}\right).
\end{aligned}$$

(c)
- When $\beta=\pi$, the two planes form an infinite large plane, and the Green function

$$G = \ln\left(\frac{\rho^2+\rho'^2-2\rho\rho'\cos(\phi+\phi')}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right) = \ln\frac{|\boldsymbol{r}-\boldsymbol{r}''|^2}{|\boldsymbol{r}-\boldsymbol{r}'|^2},$$

where $\boldsymbol{r}''$ is the location of the image charge that locates at $\boldsymbol{r}'$, is aligned with the physics.
- When $\beta=\dfrac{\pi}{2}$, the two plane is perpendicular to each other, which is in line with Problem 2.3. And the Green function is

$$\begin{aligned}
G &= \ln\frac{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi+\phi')][\rho^2+\rho'^2+2\rho\rho'\cos(\phi+\phi')]}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')][\rho^2+\rho'^2+2\rho\rho'\cos(\phi-\phi')]}\\
&= \ln\frac{|\boldsymbol{r}-\boldsymbol{r}_3|^2|\boldsymbol{r}-\boldsymbol{r}_4|^2}{|\boldsymbol{r}-\boldsymbol{r}_1|^2|\boldsymbol{r}-\boldsymbol{r}_2|^2},
\end{aligned}$$

which is aligned with the result of Problem 2.3.

26. (a) The general solution of the 2D-Poisson equation is

$$\Phi(\rho,\phi) = A_0 + B_0\ln\rho + \sum_{n=1}^{\infty}\left(A_n\rho^{n\pi/\beta}+B_n\rho^{-n\pi/\beta}\right)\left(C_n\sin\frac{n\pi\phi}{\beta}+D_n\cos\frac{n\pi\phi}{\beta}\right).$$

When $\phi=0$ or $\phi=\beta$, $\Phi=0$ for any ρ. Hence

$$A_0 = B_0 = 0, \quad D_n = 0(n=1,2,\cdots).$$

When $\rho=a$, $\Phi=0$ for any $\phi\in(0,\beta)$. Hence

$$B_n = -A_n a^{2n}.$$

Consequently, the potential can be expressed as

$$\Phi(\rho,\phi) = \sum_{n=1}^{\infty}\alpha_n\left[\left(\frac{\rho}{a}\right)^n - \left(\frac{a}{\rho}\right)^n\right]\sin\frac{n\pi\phi}{\beta},$$

where α_n is the coefficient determined by the boundary condition when $\rho\to\infty$.

(b) The lowest nonvanishing term of electric potential is

$$\Phi = \alpha_1 \left[\left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{a}{\rho}\right)^{\pi/\beta} \right] \sin\frac{\pi\phi}{\beta}.$$

Thus, the electric field is

$$E_\rho = -\frac{\alpha_1\pi}{\beta\rho} \left[\left(\frac{\rho}{a}\right)^{\pi/\beta} + \left(\frac{a}{\rho}\right)^{\pi/\beta} \right] \sin\frac{\pi\phi}{\beta},$$

$$E_\phi = -\frac{\alpha_1\pi}{\beta\rho} \left[\left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{a}{\rho}\right)^{\pi/\beta} \right] \cos\frac{\pi\phi}{\beta}.$$

And the charge density on the surface $\phi = 0$ or $\phi = \beta$ is

$$\sigma(\rho, 0) = \sigma(\rho, \beta) = \epsilon_0 E_\phi(\rho, 0) = -\frac{\pi\epsilon_0\alpha_1}{\beta\rho} \left[\left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{a}{\rho}\right)^{\pi/\beta} \right],$$

and the charge density on the surface $\rho = a$ is

$$\sigma(a, \phi) = \epsilon_0 E_r(a, \phi) = -\frac{2\pi\epsilon_0\alpha_1}{\beta a} \sin\frac{\pi\phi}{\beta}.$$

(c) When $\beta = \pi$, the electric field is

$$E_\rho = -\frac{\alpha_1}{\rho} \left(\frac{\rho}{a} + \frac{a}{\rho} \right) \sin\phi,$$

$$E_\phi = -\frac{\alpha_1}{\rho} \left(\frac{\rho}{a} - \frac{a}{\rho} \right) \cos\phi.$$

When $\phi = 0$ or $\phi = \pi$, and $\rho \to \infty$,

$$E_\rho \equiv 0,$$

$$E_\phi = \mp\frac{\alpha_1}{a},$$

which demonstrates that the electric field is $\boxed{\text{uniform and normal to the plane}}$

Hence the charge distribution is

$$\sigma(a, \phi) = -\frac{2\epsilon_0\alpha_1}{a} \sin\phi,$$

$$\sigma(\rho, 0) = \sigma(\rho, \pi) = -\epsilon_0\alpha_1 \left(\frac{1}{a} - \frac{a}{\rho^2} \right).$$

The total charge per unit length in z direction on the half-cylinder is

$$Q_1 = \int_0^\pi \sigma(a, \phi) a \mathrm{d}\phi = -4\epsilon_0\alpha_1,$$

and the total charge per unit length in z direction on the strip of width $2a$ in the absence of the half-cylinder is

$$Q_2 = \sigma(\infty, 0) \cdot 2a = -2\epsilon_0\alpha_1.$$

Hence,

$$\boxed{Q_1 = 2Q_2}$$

Suppose $b \gg a$, and then the total charge on a strip of $2b$ in the presence of the half-cylinder is

$$Q = 2\int_a^b \sigma(\rho, 0)\mathrm{d}\rho + Q_1 = -2\epsilon_0\alpha_1\left(\frac{b}{a} + \frac{a}{b}\right) \simeq -2\epsilon_0\alpha_1\frac{b}{a}.$$

On the other hand, the total charge on the same strip in the absence of the half-cylinder is

$$Q' = \sigma(\infty, 0)\cdot 2b = -2\epsilon_0\alpha_1\frac{b}{a}.$$

Hence, $Q = Q'$, which implies that the extra charge on the half-cylindrical surface comes from regions of the plane nearby.

27. (a) According to the solution of Problem 2.26, the charge distribution on the cylinder is

$$\sigma(a, \phi) = -\frac{\epsilon_0\alpha_1}{a}\sin\frac{\phi}{2},$$

and the charge distribution on the top and bottom of the sheet is

$$\sigma(\rho, 0) = \sigma(\rho, 2\pi) = -\frac{\epsilon_0\alpha_1}{2\rho}\left[\sqrt{\frac{\rho}{a}} - \sqrt{\frac{a}{\rho}}\right].$$

(b) The total charge on the cylinder is

$$Q_1 = -\frac{\epsilon_0\alpha_1}{a}\int_0^{2\pi} \sin\frac{\phi}{2} a\mathrm{d}\phi = -4\epsilon_0\alpha_1.$$

The total charge on the sheet near the cylinder is

$$Q_2 = 2\cdot\frac{\epsilon_0\alpha_1}{2}\int_a^b \frac{1}{\rho}\left[\sqrt{\frac{\rho}{a}} - \sqrt{\frac{a}{\rho}}\right]\mathrm{d}\rho = -2\epsilon_0\alpha_1\left[\sqrt{\frac{\rho}{a}} + \sqrt{\frac{a}{\rho}} - 2\right]$$

Suppose $\sigma(b, 0) \simeq -\dfrac{\epsilon_0\alpha_1}{2\sqrt{ab}} (b \gg a)$ is a constant σ_0, then

$$\alpha_1 = -\frac{2\sigma_0\sqrt{ab}}{\epsilon_0}.$$

Consequently,

$$Q_2 = 4\sigma_0\left(a + b - 2\sqrt{ab}\right).$$

When $a = 0$, $Q_2' = 2\sigma_0 b$, and the deficiency is

$$\Delta Q_2 = Q' - Q = 2\sigma_0(2\sqrt{ab} - a) \simeq 8\sigma_0\sqrt{ab} = -4\epsilon_0\alpha_1,$$

which is same as the total charge on the cylinder.

28. *Green Theorem* gives:

$$\int_V (\Phi\nabla^2\Psi - \Psi\nabla^2\Phi)\mathrm{d}\tau = \oint_S \left(\Phi\frac{\partial\Psi}{\partial n'} - \Psi\frac{\partial\Phi}{\partial n'}\right)\mathrm{d}S'.$$

Let $\Psi = G(\boldsymbol{r}, \boldsymbol{r}')$, and then the left hand side is

$$-4\pi\int_V \Phi\delta(\boldsymbol{r} - \boldsymbol{r}')\mathrm{d}\tau,$$

and the right hand side is

$$\oint_S \Phi\frac{\partial G}{\partial n'}\mathrm{d}S'.$$

Hence,

$$\Phi(O) = -\frac{1}{4\pi}\oint_S \Phi\frac{\partial G}{\partial n'}\mathrm{d}S' - \frac{1}{4\pi}\sum_{i=1}^{n}\int_{S_i}\Phi_i\frac{\partial G}{\partial n'}\mathrm{d}S',$$

where O is the center point. Because of the symmetry of the configuration,

$$\int_{S_i}\frac{\partial G}{\partial n'}\mathrm{d}S'$$

are equal. Thus,

$$\Phi(O) = -\frac{1}{4\pi}\sum_{i=1}^{n}\Phi_i\int_{S_1}\frac{\partial G}{\partial n'}\mathrm{d}S'.$$

Let $\Phi_i = V(i = 1, 2, \cdots)$, and then $\Phi(O) = V$,

$$V = -\frac{nV}{4\pi}\int_{S_1}\frac{\partial G}{\partial n'}\mathrm{d}S'.$$

Hence

$$\int_{S_1}\frac{\partial G}{\partial n'}\mathrm{d}S' = -\frac{4\pi}{n}.$$

Consequently,

$$\Phi(O) = \frac{1}{n}\sum_{i=1}^{n}\Phi_i.$$

29. Omitted. (The relation is too easy to verify!)

30. In this case, $\psi_1 = \psi_{1,1} = \psi_{1,3} = \psi_{3,1} = \psi_{3,3}$, $\psi_2 = \psi_{1,2} = \psi_{2,1} = \psi_{2,3} = \psi_{3,2}$, $\psi_3 = \psi_{2,2}$. A linear inhomogeneous equation with regard to ψ_1, ψ_2, ψ_3 can be derived from the integral:

$$\begin{cases} \frac{8}{3}\psi_1 - \frac{2}{3}\psi_2 - \frac{1}{3}\psi_3 = \frac{\pi}{4}, \\ -\frac{2}{3}\psi_1 + 2\psi_2 - \frac{1}{3}\psi_3 = \frac{\pi}{4}, \\ -\frac{4}{3}\psi_1 - \frac{4}{3}\psi_2 + \frac{8}{3}\psi_3 = \frac{\pi}{4}. \end{cases}$$

Hence,

$$\begin{cases} \psi_1 = 0.609, \\ \psi_2 = 0.757, \\ \psi_3 = 0.976. \end{cases}$$

Compared with the exact values, this solution has the relative values:

$$\begin{cases} e_1 = 7.03\%, \\ e_2 = 4.99\%, \\ e_3 = 5.40\%, \end{cases}$$

which demonstrates that our solution is close to the exact values to some extent. And the accuracy can be improved if h is smaller.

3 Boundary-Value Problems in Electrostatics:II

1. The general solution in spherical coordinate system is

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\left(A_l r^l+B_l r^{-l-1}\right)P_l(\cos\theta).$$

The boundary condition

$$\Phi(a,\theta)=\begin{cases}V, & \theta<\dfrac{\pi}{2},\\ 0, & \theta>\dfrac{\pi}{2},\end{cases}\quad \Phi(b,\theta)=\begin{cases}0, & \theta<\dfrac{\pi}{2},\\ V, & \theta>\dfrac{\pi}{2}\end{cases}$$

dictates that

$$A_l a^l+B_l a^{-l-1}=\frac{2l+1}{2}\int_0^{\pi}\Phi P_l(\cos\theta)\sin\theta\mathrm{d}\theta\overset{x=\cos\theta}{=\!=\!=}\frac{2l+1}{2}V\int_0^1 P_l(x)\mathrm{d}x,$$

$$A_l b^l+B_l b^{-l-1}=\frac{2l+1}{2}\int_0^{\pi}\Phi P_l(\cos\theta)\sin\theta\mathrm{d}\theta\overset{x=\cos\theta}{=\!=\!=}\frac{2l+1}{2}V\int_{-1}^0 P_l(x)\mathrm{d}x.$$

- When $l=0$,

$$\begin{cases}A_0+B_0a^{-1}=V/2,\\ A_0+B_0b^{-1}=V/2.\end{cases}$$

Hence

$$A_0=\frac{V}{2},\quad B_0=0.$$

- When $l\neq 0$, as Legendre Polynomial has the recurrence relation

$$P'_{n+1}(x)-P'_{n-1}(x)=(2n+1)P_n(x),$$

$$\int_0^1 P_l(x)\mathrm{d}x=(-1)^l\int_{-1}^0 P_l(x)\mathrm{d}x=\frac{1}{2l+1}\left(P_{l+1}(x)-P_{l-1}(x)\right)\Big|_0^1=\frac{1}{2l+1}\left(P_{l-1}(0)-P_{l+1}(0)\right).$$

If $l=2k$ is even,

$$P_{l-1}(0)-P_{l+1}(0)=0.$$

Then

$$A_{2k}=B_{2k}=0.$$

If $l=2k+1$ is odd,

$$P_{l-1}-P_{l+1}=(-1)^{\frac{l-1}{2}}\frac{(l-2)!!}{(l-1)!!}-(-1)^{\frac{l+1}{2}}\frac{l!!}{(l+1)!!}=(-1)^k\frac{(2k-1)!!}{(2k)!!}\frac{4k+3}{2k+2},$$

Let $C_{2k+1}=\dfrac{V}{2}\left(P_{l-1}(0)-P_{l+1}(0)\right)$, and then

$$A_{2k+1}a^{2k+1}+B_{2k+1}b^{-2k-2}=C_{2k+1},$$

$$A_{2k+1}b^{2k+1}+B_{2k+1}b^{-2k-2}=-C_{2k+1}.$$

Consequently, we have

$$A_{2k+1}=-\frac{C_{2k+1}}{b^{2k+1}}\frac{\left[1+\left(\frac{a}{b}\right)^{2k+2}\right]}{\left[1-\left(\frac{a}{b}\right)^{4k+3}\right]},\quad B_{2k+1}=C_{2k+1}a^{2k+2}\frac{\left[1+\left(\frac{a}{b}\right)^{2k+1}\right]}{\left[1-\left(\frac{a}{b}\right)^{4k+3}\right]}.$$

To summary,

$$\Phi(r,\theta,\phi)=\frac{V}{2}+$$
$$\frac{V}{2}\sum_{k=0}^{\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}\frac{4k+3}{2k+2}\frac{P_{2k+1}(\cos\theta)}{\left[1-\left(\frac{a}{b}\right)^{4k+3}\right]}\left[\left(1+\left(\frac{a}{b}\right)^{2k+1}\right)\left(\frac{a}{r}\right)^{2k+2}-\left(1+\left(\frac{a}{b}\right)^{2k+2}\right)\left(\frac{r}{b}\right)^{2k+1}\right].$$

When $a\to 0$,

$$\Phi=\frac{V}{2}-\frac{V}{2}\sum_{k=0}^{\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}\frac{4k+3}{2k+2}\left(\frac{r}{b}\right)^{2k+1}P_{2k+1}(\cos\theta).$$

When $b\to\infty$,

$$\Phi=\frac{V}{2}+\frac{V}{2}\sum_{k=0}^{\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}\frac{4k+3}{2k+2}\left(\frac{a}{r}\right)^{2k+2}P_{2k+1}(\cos\theta).$$

To check the solution, we can consider the problem in Sec.3.3. That case can be seen as the superposition of a sphere with constant boundary potential V and a sphere with boundary condition

$$\Phi=\begin{cases}0, & \theta<\dfrac{\pi}{2},\\ -2V, & \theta>\dfrac{\pi}{2}.\end{cases}$$

It is easy to verify Eq.(3.36) by means of the solution of this problem.

2. (a) The potential inside the sphere satisfies

$$\begin{cases}\nabla^2\Phi_{\text{in}}=0,\\ \Phi_{\text{in}}\text{ is finite}, r\to 0.\end{cases}$$

The potential outside the sphere satisfies

$$\begin{cases}\nabla^2\Phi_{\text{out}}=0,\\ \Phi_{\text{out}}\to 0, r\to\infty.\end{cases}$$

Hence, the formal solutions are

$$\Phi_{\text{in}}=\sum_{l=0}^{\infty}A_lP_l(\cos\theta)r^l,$$

$$\Phi_{\text{out}}=\sum_{l=0}^{\infty}B_lP_l(\cos\theta)r^{-l-1}.$$

Meantime, the boundary condition on the sphere is

$$\Phi_{\text{in}}(R)=\Phi_{\text{out}}(R),$$

$$\left(\frac{\partial\Phi_{\text{in}}}{\partial r}-\frac{\partial\Phi_{\text{out}}}{\partial r}\right)_{r=R}=\sigma(\theta)=\begin{cases}0, & \theta<\alpha,\\ \dfrac{Q}{4\pi\epsilon_0R^2}, & \theta>\alpha.\end{cases}$$

Hence,

$$\left(\frac{\partial\Phi_{\text{in}}}{\partial r}-\frac{\partial\Phi_{\text{out}}}{\partial r}\right)_{r=R}=\frac{B_0}{R^2}+\sum_{l=1}^{\infty}P_l(\cos\theta)\left(lA_lR^{l-1}+(l+1)B_lR^{-l-2}\right),$$

where

$$\frac{B_0}{R^2} = \int_\alpha^\pi \sigma(\theta) P_0(\cos\theta)\sin\theta \mathrm{d}\theta = \frac{Q}{4\pi\epsilon_0 R^2}(\cos\alpha + 1),$$

$$A_0 = \frac{B_0}{R} = \frac{Q}{4\pi\epsilon_0 R}(\cos\alpha + 1),$$

and

$$lA_l R^{l-1} + (l+1)B_l R^{-l-2} = \int_\alpha^\pi P_l(\cos\theta)\sigma(\theta)\sin\theta \mathrm{d}\theta = \frac{Q}{4\pi\epsilon_0}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right],$$

$$A_l R^l = B_l R^{-l-1},$$

i.e.

$$A_l = \frac{Q}{8\pi\epsilon_0 R^{l+1}(2l+1)}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right],$$

$$B_l = \frac{QR^l}{8\pi\epsilon_0(2l+1)}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right].$$

In conclusion,

$$\Phi_{\text{in}} = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta),$$

$$\Phi_{\text{out}} = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\frac{R^l}{r^{l+1}}P_l(\cos\theta).$$

(b) For the term $l = 0$, Φ_0 is a constant. Hence, it contributes nothing to the electric field. For the term $l = 1$,

$$\Phi_1 = \frac{Q}{8\pi\epsilon_0}\frac{1}{3}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]\frac{r\cos\theta}{R^2} = -\frac{Qz}{16\pi\epsilon_0 R^2}\sin^2\alpha.$$

Hence, the electric field contributed by Φ_1 is

$$\boldsymbol{E}_1 = -\nabla\Phi_1 = \frac{Q\sin^2\alpha}{16\pi\epsilon_0 R^2}\hat{\boldsymbol{z}}.$$

For the terms $\Phi_l (l \geq 2)$, $\Phi_l \sim r^l$, and thus they contribute nothing to the electric field at the origin. In conclusion, the electric field at the origin is

$$\boldsymbol{E} = \frac{Q\sin^2\alpha}{16\pi\epsilon_0 R^2}\hat{\boldsymbol{z}}.$$

(c) (1) $\alpha \to 0$: All terms vanish except $l = 0$, and hence the potential is

$$\Phi = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 R}, & r < R, \\ \dfrac{Q}{4\pi\epsilon_0 r}, & r > R. \end{cases}$$

The electric field at the origin

$$\boldsymbol{E} = 0.$$

Apparently, all the results are aligned with what we already know.

(2) $\alpha \to \pi$: $P_l(-1) = (-1)^l$ indicates that all terms vanish and hence the potential

$$\Phi = 0.$$

The electric field is

$$\boldsymbol{E} = 0.$$

3. Suppose the charge density on the disc is

$$\sigma = \frac{\sigma_0}{\sqrt{R^2 - \rho^2}},$$

where σ_0 is a constant. Then the potential along z axis is

$$\Phi(z) = \frac{\sigma_0}{2\epsilon_0}\int_0^R \frac{\rho \mathrm{d}\rho}{\sqrt{(z^2+\rho^2)(R^2-\rho^2)}} = \frac{\sigma_0}{4\epsilon_0}\left(\frac{\pi}{2} - \arcsin\frac{z^2-R^2}{z^2+R^2}\right).$$

Hence $\Phi(0) = V$ gives:

$$\sigma_0 = \frac{4\epsilon_0 V}{\pi},$$

and the final solution of potential along z axis is:

$$\Phi(z) = \frac{V}{\pi}\left(\frac{\pi}{2} - \arcsin\frac{z^2-R^2}{z^2+R^2}\right). \qquad (*)$$

(a) When $z > R$, Eq.(*) can be expanded as

$$\Phi(z) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l+1}.$$

On the other hand, the general solution for the case $r > R$ should be

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} A_l r^{-(l+1)} P_l(\cos\theta),$$

where the coefficients of r^l vanish because $\Phi = 0$ when $r \to \infty$. By comparing the general solution and the special solution along z axis, we have

$$A_{2l} = \frac{2V}{\pi}\frac{(-1)^l}{2l+1}R^{2l+1}, \quad A_{2l+1} = 0, \quad l = 0, 1, 2, \cdots.$$

Consequently,

$$\boxed{\Phi(r,\theta,\phi) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l+1} P_{2l}(\cos\theta)}$$

(b) When $z < R$, Eq.(*) can be expanded as

$$\Phi(z) = V - \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{z}{R}\right)^{2l+1},$$

while the general solution is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} B_l r^l P_l(\cos\theta),$$

where the coefficients of $r^{-(l+1)}$ vanish because Φ is finite when $r \to 0$. Hence

$$B_0 = V, \quad B_{2l} = 0, \quad B_{2l+1} = -\frac{2V}{\pi}\frac{(-1)^l}{2l+1}\frac{1}{R^{2l+1}}, \quad l = 1, 2, \cdots.$$

Consequently,

$$\boxed{\Phi(r,\theta,\phi) = V - \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R}\right)^{2l+1} P_{2l+1}(\cos\theta)}$$

(c) The total charge is

$$Q = \int_0^R \sigma(\rho) \cdot 2\pi\rho \mathrm{d}\rho = 8\epsilon_0 RV.$$

Hence the capacitor of the disc is

$$\boxed{C = \frac{Q}{V} = 8\epsilon_0 R}$$

4. (a) The general solution of electric potential is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} r^l e^{\mathrm{i}m\phi} P_l^m(\cos\theta),$$

where the boundary condition gives:

$$\Phi(\phi) = \begin{cases} +V, & k\dfrac{2\pi}{n} \le \phi \le \left(k+\frac{1}{2}\right)\frac{2\pi}{n}, \\ -V, & \left(k+\frac{1}{2}\right)\frac{2\pi}{n} \le \phi (k+1)\frac{2\pi}{n}. \end{cases}$$

Hence,

$$A_{lm} = \frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}\int_0^{\pi} P_l^m(\cos\theta)\sin\theta \mathrm{d}\theta \int_0^{2\pi} \Phi(\phi) e^{\mathrm{i}m\phi}\mathrm{d}\phi.$$

5. The Green function of this spherical system

$$G(\boldsymbol{r},\boldsymbol{r'}) = \frac{1}{\left(r^2+r'^2-2rr'\cos\gamma\right)^{1/2}} - \frac{a/r'}{\left(r^2+(a^2/r')^2-2(a^2/r')r\cos\gamma\right)^{1/2}}$$

is given by the image method. Then

$$\left.\frac{\partial G}{\partial n'}\right|_{\text{sphere}} = \left.\frac{\partial G}{\partial r'}\right|_{r'=a} = \frac{(r^2-a^2)/a}{(r^2+a^2-2ar\cos\gamma)^{3/2}}.$$

Consequently, the potential inside the sphere is

$$\Phi(\boldsymbol{r}) = -\frac{1}{4\pi}\oint_{\text{sphere}} V(\theta',\phi')\frac{\partial G}{\partial n'}\mathrm{d}S' = \frac{a(a^2-r^2)}{4\pi}\oint_{\text{sphere}} \frac{V(\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{1/2}}\mathrm{d}\Omega'.$$

On the other hand, the potential inside the sphere satisfies *Laplace's equation*, and the general solution is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} r^l Y_{lm}(\theta,\phi),$$

where the coefficient of $r^{-(l+1)}$ vanishes because the potential is finite when $r \to 0$. Hence,

$$V(\theta',\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} a^l Y_{lm}(\theta',\phi'),$$

and the coefficients

$$A_{lm} = \frac{1}{a^l}\int V(\theta',\phi') Y_{lm}(\theta',\phi')\mathrm{d}\Omega$$

can be derived by the orthogonality of Y_{lm}.

As you can see, these two ways lead to the respective results while they describe the same physical system, so these two results are equivalent.